Sporadic Groups: Exceptions, or Part of a Pattern?

Last month news of the apparent discovery of yet another sporadic group filtered through the mathematical community. If confirmed, this mathematical entity would be one of only 21 known examples of the rare and still poorly understood class of finite groups known as sporadic. The discovery would also be the sixteenth such group found since 1965, reflecting a burst of activity in finite group theory that has turned up new sporadic groups almost every year and has resulted in several other important developments in the past decade. Among the most intriguing of these developments, although its importance is still uncertain, is the demonstration of an intimate connection between some sporadic groups and error-correcting codes of the type used to transmit binary information reliably in distortion-prone environments.

Sporadic groups play something of the role of the joker in finite group theory, which has as a basic goal determination of the structure of all finite groups. In classifying these groups mathematicians employ the concept of a simple group, one which, in some sense, cannot be decomposed into subgroups. Simple groups thus constitute the building blocks out of which all finite groups are constructed. There are two well-known classes of simple groups: alternating groups, consisting of all even permutations of n objects, and groups of the Lie type, which have strong geometrical analogies. Both classes have an infinite number of simple groups (each of which has a finite number of elements), but these groups occur in regular patterns and their properties have been extensively studied. Most simple groups belong to one of these two classes. The major exceptions are the sporadic groups.

Sporadic groups, as their name implies, do not seem to follow any regular pattern. If there should turn out to be an infinite number of such anomalous groups, then hopes of classifying all finite groups would have to be discarded. Hence, there is considerable interest in sporadic groups and in the possibility that they are either limited in number or ordered in some way not yet perceived.

Most of the known sporadic groups were found almost by accident in the course of trying to establish basic theorems about finite groups. Considering the uncertainties still surrounding sporadic groups, some remarkably general theorems have been proved. In 1963, for example, J. Thompson, then at the University of Chicago, and W. Feit of Yale showed that essentially every simple group has an even number of elements. Other investigators, trying to prove similar theorems, have on occasion run into difficulties that, on further examination, turned out to be evidence of a new sporadic group.

The most recently discovered sporadic group was found by M. O'Nan of Rutgers University. He was trying to classify a particular family of permutation groups. In the process, he encountered a configuration that could not be explained in terms of known simple groups or in terms of general theorems. O'Nan worked out the struc-

(Continued on page 148)

Mathematical Groups

One of the simplest and most useful algebraic structures is the group, study of which dates back to 1830. The concept is not only of mathematical interest, but has found application in fields ranging from quantum mechanics to crystallography. Despite their long history and many uses, groups remain a remote concept to many. Since the accompanying two articles concern advances in group theory and its applications, this short summary of the basic axioms and properties of groups is offered as a convenient primer.

A group is a set of elements together with an operation which satisfies the four axioms shown in Table 1. The elements (denoted **a**, **b**, . . .) may be any sort of object or transformation, including numbers, vectors, physical motions, and geometric spaces. The operation (denoted by *) may be algebraic or geometric, including addition, matrix multiplication, and rotation. In some instances what is of interest is a semi-group (a set of elements that obey all but one of the axioms; a semi-group may not have inverse elements, for example.

A familiar example of a group is the set of all integers (. . .—1, 0, 1, 2, . . .) combined with the operation of addition. Zero is the identity element and the inverse for an integer is its negative.

Table 1. Group axioms.

Closure:	for any elements $\mathbf{a},\mathbf{b}$ $\mathbf{a}*\mathbf{b}$ is an element of the group
Associative:	$(\mathbf{a}*\mathbf{b})*\mathbf{c} = \mathbf{a}*(\mathbf{b}*\mathbf{c})$
Identity:	there is an element $\mathbf{I}$ such that $\mathbf{I}*\mathbf{a} = \mathbf{a}*\mathbf{I} = \mathbf{a}$
Inverse:	for every element $\mathbf{a}$ there is an element $\mathbf{a}^{-1}$ such that $\mathbf{a}*\mathbf{a}^{-1} = \mathbf{a}^{-1}*\mathbf{a} = \mathbf{I}$

In this case, there is an infinite number of elements in the group.

Another common group is the set of all (nonsingular) n by n matrices with matrix multiplication as the group operation—the group known as the full linear group of dimension n. Not only is there an infinite number of elements in each group, but there are an infinite number of such groups.

An example of a finite group is given in Fig. 1. The elements of the group are the possible ways of rotating a square so as to change the relative orientation of its vertices from one of the eight possible positions to another; these eight motions, including the option of no motion at all (the identity element), comprise the group. Combination of two successive rotations, for example **a** followed by **d**, is the group operation ($\mathbf{a}*\mathbf{d}$ is equivalent to **g**, another element of the group, as required by the closure axiom). For this paricular group, the operation is not commutative and the order in which it is applied makes a difference ($\mathbf{a}*\mathbf{d} = \mathbf{g} \neq \mathbf{d}*\mathbf{a}$). Inverse elements exist for all members of the group—$\mathbf{d}*\mathbf{d} = \mathbf{I}$, for example.

The operations for a group may be summarized in a group multiplication

Phase Changes: A Universal Theory of Critical Phenomena

In the last 2 years a sweeping new theory of critical phenomena has been proposed. Although primarily a result of new physical insight, the advance involves group theory in some forms of its expression.

Everyone knows that when water boils at 100°C, a dramatic change in the density of the liquid takes place as it turns to gas. But when water boils at higher temperatures, as it must if the pressure is greater than atmospheric pressure, the change of density that occurs in the liquid-to-gas transition decreases until at a sharply defined critical temperature it disappears entirely. Above the temperature of 374°C, water exists only as steam. All liquid-gas systems behave in a similar way, though the critical temperatures vary widely from one substance to another.

But liquid-gas systems are examples of only one type of critical phenomenon. Liquid solutions, biopolymers, superfluids, liquid crystals, alloys, superconductors, and ferromagnetic metals all undergo phase transitions that are also classed as critical phenomena. If a nickel magnet is heated, its magnetization decreases until it abruptly vanishes at 354°C, the critical temperature. At room temperature the crystal structure of a brass alloy composed of equal numbers of copper and zinc atoms is very regular, with atoms of the two metals located at alternate sites in a cubic lattice. But when the alloy is heated, the pattern becomes less regular, as measured by x-ray or neutron scattering, until it is completely destroyed at 466°C. Yet another example of a critical phase transition occurs in liquid helium. Below the critical temperature of −271.0°C, helium can exist in a superfluid phase. But above that temperature the superfluid, which has many unusual properties, disappears.

By about 1965 it was clear that the classical theories of phase transitions were inadequate to describe critical phenomena, and in the following years many theorists began to realize that different classes of critical behavior are related in ways that are essentially independent of the physical details of the different systems. While the critical temperatures of various systems depend on specific physical details, such as the strength of a molecular force or the interatomic spacing in a lattice, the qualitative aspects of critical behavior are apparently independent of those details and constant, not only within classes of critical phenomena, but also from one class to another.

For example, the difference in density between liquid and gas—for water or any other liquid—decreases as the temperature approaches the critical temperature (T_c) with the particular functional dependence $(T - T_c)^{\beta}$. The exponent β has been measured to be very nearly ⅓. The magnetization of nickel and the degree of order in brass approach zero with the same qualitative dependence on temperature, and again β is nearly ⅓. Furthermore, β is only one of the so-called critical exponents that seem to have the same values for many different classes of critical phenomena. Besides an exponent to describe the coexistence of different components, the complete

(Continued on page 149)

table such as that given in Fig. 1. Often it is convenient to think of groups not as composed of particular objects such as numbers or rotations, but as a set of abstract elements subject to the combination rule expressed by a multiplication table. In this more general view, groups with the same number of elements and the same multiplication table are essentially identical. The enumeration of all abstract groups—which have been compared to the grin that remains when the Cheshire cat fades away—and the determination of their properties are the basic tasks of group theory.

—A.L.H.

I Identity: Don't move square

a Rotate square 90° counter-clockwise

b Rotate square 180° counter-clockwise

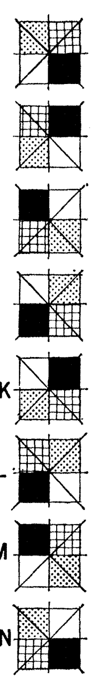
c Rotate square 270° counter-clockwise

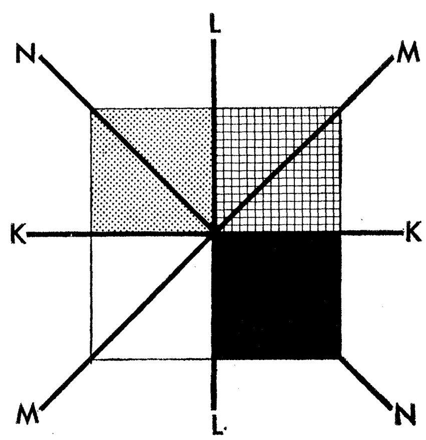
d Flip over square around axis K-K

e Flip over square around axis L-L

f Flip over square around axis M-M

g Flip over square around axis N-N





Multiplication table

*	I	a	b	c	d	e	f	g
I	I	a	b	c	d	e	f	g
a	a	b	c	I	g	f	d	e
b	b	c	I	a	e	d	g	f
c	c	I	a	b	f	g	e	d
d	d	f	e	g	I	b	a	c
e	e	g	d	f	b	I	c	a
f	f	e	g	d	c	a	I	b
g	g	d	f	e	a	c	b	I

Example: $a * d = g$

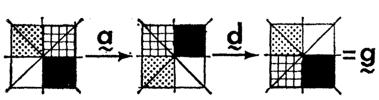


Fig. 1. A finite group with eight elements. [Kenneth Smith]