and the incoherent surface energy of the interfaces of the twin is neglected. The incremental work done by the external stress on altering the twin dimension in the shear direction χ_1 may be expressed in terms of the number of partial dislocations n composing the twin as

$$\delta W_c \approx b_1 \sigma_{12} n (d\chi_1 \times dl_3) \qquad (3)$$

From Eq. 2, the force dF_1 acting to produce this displacement is given by

$$dF_1 \approx n b_1 \sigma_{12} \times dl_3 \qquad (4)$$

Since

$$n b_1 = n \gamma_{12} b_2 = \gamma_{12} t$$

where t is the twin thickness, the force per unit length acting on the twin is

$$(dF_1/dl_3) \approx \gamma_{12} t \, \sigma_{12} \qquad (5)$$

If it is presumed that a critical resolved shear stress for deformation twinning τ_{CRTS} exists by virtue of a constant force per unit length acting on the twin, as has been argued for the slip process (*4*), then τ_{CRTS} is inversely proportional to the twin thickness.

Thus the invariant twinning shear is the essential feature of a deformation twin. This feature makes the second type of dislocation (Eq. 1 and Fig. 1) useful for developing a more complete model for deformation twinning. The simple shear displacements associated with the wedge dislocation (Fig. 1) may be derived from the equations of equilibrium for this body, because a simple shear is composed of a rotation plus a pure shear (*5*). If the terms resulting from the conditions at the surface boundary are neglected

$$u_1 = (d_{12}/2\pi) \{ -\chi_2 \tan^{-1} (\chi_1/\chi_2) - [(1-2\nu)/4(1-\nu)] \chi_1 \ln (\chi_1^2 + \chi_2^2)\} \qquad (6)$$

where ν is Poissons ratio. The terms in Eq. 6 are a multiple combination of several of those in the complete solutions of the displacements for a single edge dislocation (*6*). Figure 3 shows the simple shear displacements which produce the twin. Further properties of a deformation twin may be directly determined through a complete analysis of these wedge dislocations.

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Mass of Pluto

Abstract. *Analysis of the observations of Neptune indicates a reciprocal mass of Pluto of 1,812,000 (0.18 Earth masses). If the density is the same as that of Earth, the diameter would be 7200 kilometers. If 6400 kilometers is accepted (from other sources) as the upper limit of the diameter, then Pluto must be at least 1.4 times as dense as Earth.*

One of the outstanding discordances among the solar system constants is the inconsistency between the physically measured diameter of Pluto and the dynamical determination of the mass of Pluto from its perturbation of the motions of Neptune and Uranus. Measurement of the disk of Pluto by Kuiper (*1*), using the 200-inch telescope, revealed an apparent semidiameter of 0.23 arc sec with an internal consistency of ± 0.01 arc sec. Use of the adopted value of the astronomical unit in kilometers, leads to a value of the diameter of Pluto of 5928 km. More recently, from a near occultation of a 15th-magnitude star by Pluto, Halliday *et al.* (*2*) determined an upper limit to the diameter of Pluto of 6400 km. If we use the direct measurement, or the upper limit, for the diameter of Pluto and then assume that the density of Pluto does not exceed the density of Earth, the corresponding values for the mass of Pluto would be 0.10 or 0.13 Earth masses, respectively. If, on the other hand, the dynamical determination of the mass of Pluto by Wylie (*3*) of 0.91 Earth masses (Sun/Pluto = 360,000), based on an analysis of the motion of Neptune, or the determination by Brouwer (*4*) of 0.82 Earth masses, based on the motion of both Uranus and Neptune, is utilized in combination with the above measurements of the diameter, the mean density of Pluto would have to be at least 40 g/cm^3.

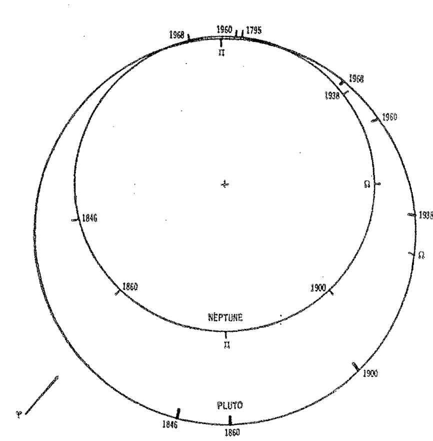

Fig. 1. Two-dimensional view of the orbits of Neptune and Pluto. Ω, node; Π, perihelion; γ, vernal equinox.

The discovery of Neptune in 1846 was one of the triumphs of celestial mechanics. Both Leverrier and Adams, on the basis of the departure of observations of Uranus from gravitational theory, were able to predict the presence and location of Neptune. Although the presence of a trans-Neptunian planet was long suspected, Wylie's analysis (*3*) has shown that its location could not be predicted gravitationally. The discovery of Pluto in 1930 must be considered as being due more to an intensive astrometric search than to any prior knowledge of position from gravitational theory.

The orbits of Neptune and Pluto form an interesting system. As shown in Fig. 1, it appears that the orbit of Pluto actually crosses the orbit of Neptune near perihelion, but, while Neptune's orbit lies principally in the plane of the ecliptic, the orbit of Pluto is inclined to this plane by 17°. An analysis of the motions of these two planets over an extended period of time (*5*) has shown that the closest approaches of the two planets librate about the aphelion of Pluto in an arc of some 76°, with a libration half-period of 10,000 years. The positions in orbit occupied by both Neptune and Pluto since discovery of Neptune are shown in Fig. 1; the point of closest approach of the two bodies occurred in 1896— at a distance of 18.9 A.U. Shown also are the nodes of the orbits of Neptune and Pluto on the ecliptic, as well as the position of Neptune in 1795 when it was observed but not recognized as a planet. These observations were later recovered and reduced by Lalande. The observations from the discovery of Neptune in 1846 to the present encompass more than 70 percent of the orbit of Neptune. Although the observations of 1795, prior to discovery, played an important role in the early determinations of the orbit, Newcomb pointed out (*6*) ". . . in the 20th century the observations made after 1846 will enable astronomers to compute the position of the planet in 1795 with a much higher degree of accuracy than Lalande could observe it." Further, since the differing treatment of the instrumental errors by Lalande and Newcomb in discussing these observations creates a corresponding difference in the observed orbital longitude of Neptune of over 7 arc sec, we have decided to omit Lalande's observations from our solution completely.

An interesting and significant characteristic of the successive theories of the motion of Neptune has been that the motion in longitude—namely, the angular position in the plane of the orbit—was not represented very far beyond the arc of observations to which the theory was originally adjusted. Newcomb's theory (*7*)—even when corrected for the presence of Pluto with

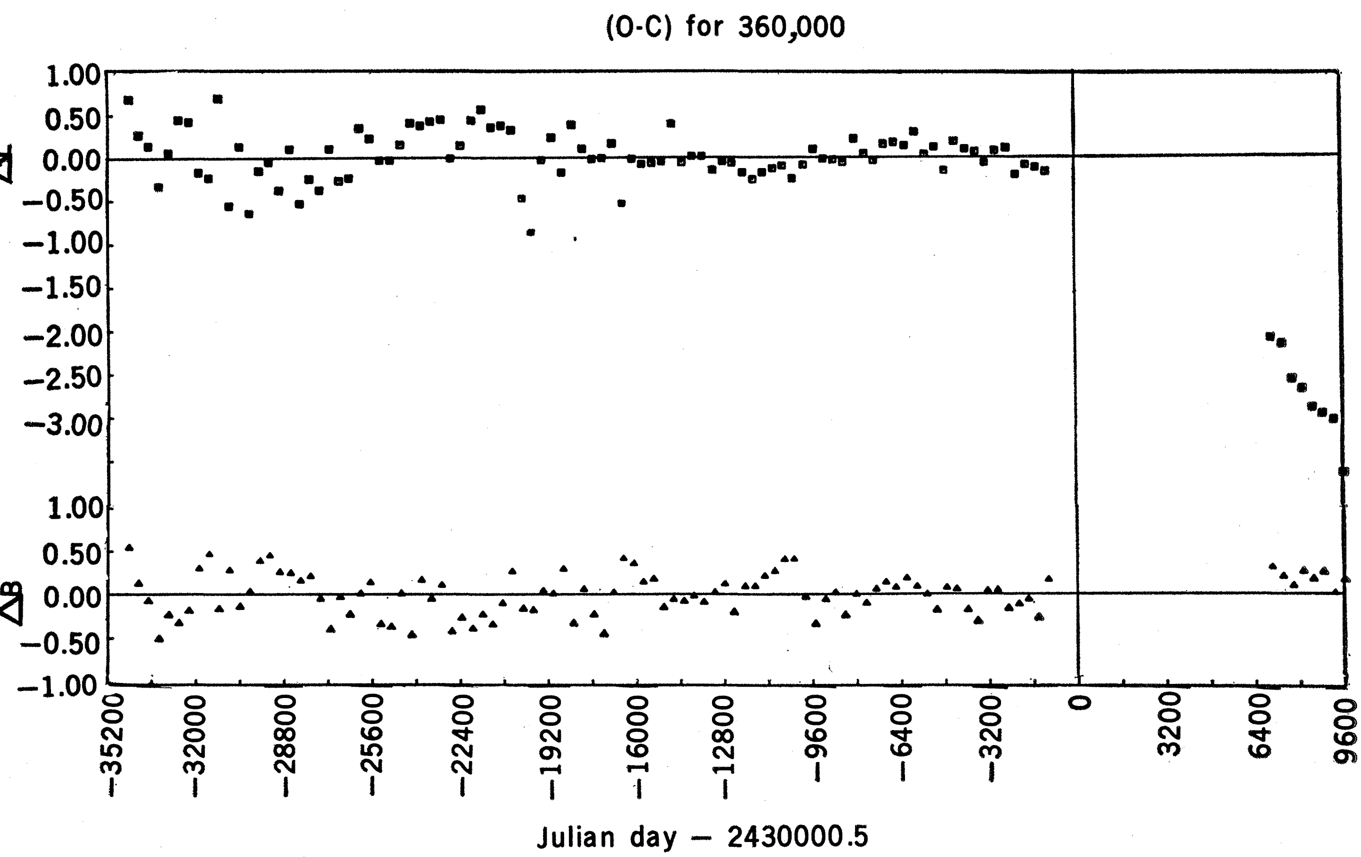

Fig. 2. The observed values minus the computed values (O-C's) in orbital longitude and latitude for the currently adopted reciprocal mass of Pluto.

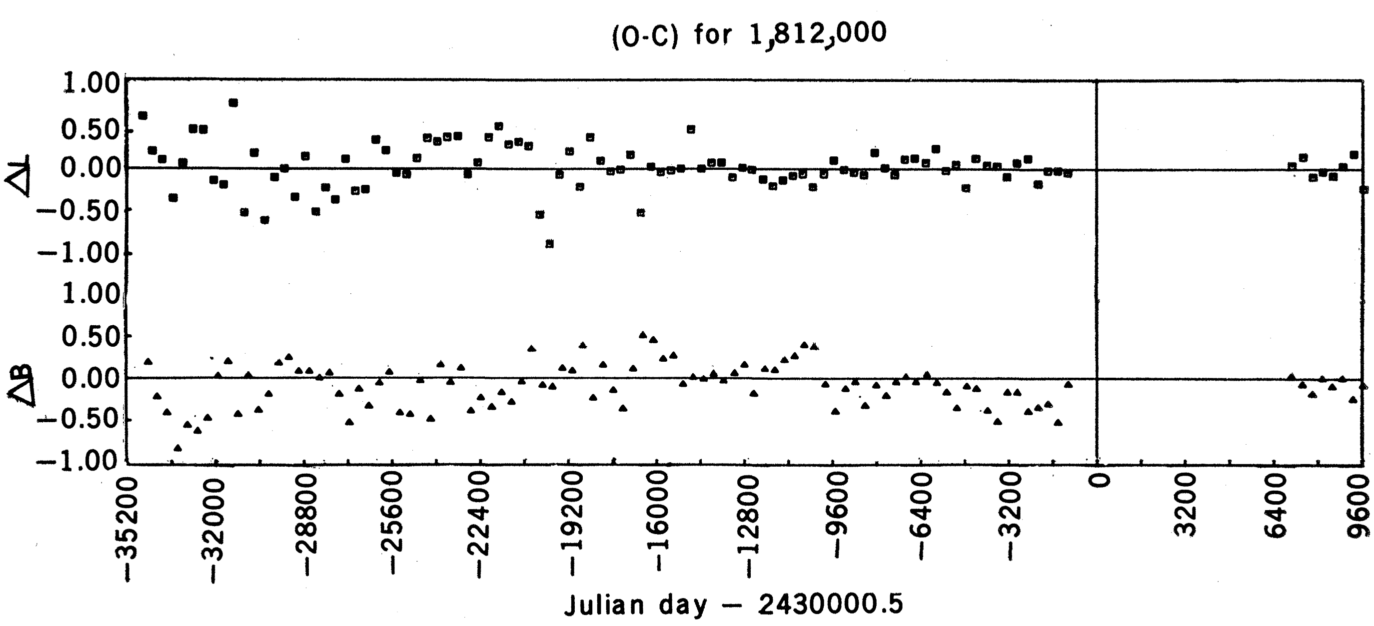

Fig. 3. Observed minus computed values (O-C's) in orbital longitude and latitude for the new reciprocal mass of Pluto.

Table 1. The sums of the squares of the residuals in orbital longitude and latitude calculated with different values for the mass of Pluto.

Reciprocal mass	Σv^2 (arc sec)					
	1846 to 1938		1960 to 1968		Total	
	$\Delta\lambda$	$\Delta\beta$	$\Delta\lambda$	$\Delta\beta$	$\Delta\lambda$	$\Delta\beta$
360,000	32.01	21.87	64.47	0.33	96.48	22.20
930,000	30.72	24.00	3.71	0.06	34.43	24.06
1,500,000	30.42	25.39	0.29	0.09	30.71	25.48
2,640,000	30.22	26.51	0.51	0.14	30.73	26.65
1,812,000	30.20	25.77	0.12	0.11	30.32	25.88

reciprocal mass of 360,000—fails to represent the position of Neptune beyond the period of observations (1795, 1848 to 1896) to which the theory was adjusted, the residuals by 1938 reaching over 5 arc sec in orbital longitude. The modern numerical theory, generated by Eckert, Brouwer, and Clemence (*8*) and incorporating the reciprocal mass 360,000 for Pluto was adjusted to fit observations through 1938, including the observations of 1795. At the present epoch, this theory fails to represent the observed longitude of Neptune by nearly 4 arc sec as shown by 158 meridian circle observations in the period 1960 to 1968. These observations referred to the FK3 system were made available to us by the Six-Inch Transit Circle Division, U.S. Naval Observatory (*9*).

Figure 2 illustrates the observed minus computed positions (O − C's) in orbital longitude and latitude compared to the Eckert, Brouwer, Clemence theory. The failure of both the earlier theory of Newcomb and the present theory of Eckert, Brouwer, and Clemence to represent the motion of Neptune in longitude apparently stems from an inability to simultaneously determine corrections to the mass of Pluto and the elliptic elements of Neptune from the span of observations available. This led us to make several simultaneous numerical integrations of the orbits of the five outer planets, with trial values of the reciprocal mass of Pluto as shown in Table 1, the first mass being that adopted in the Eckert, Brouwer, Clemence theory. With each value of the mass of Pluto, the orbit of Neptune was adjusted to fit the observations in the period 1846 to 1938, the elements of the other planets being fixed at the epoch. The integration process was repeated in order to ensure that the orbit of Neptune represented the observations. Shown in Table 1 are the sums of the squares of the residuals (Σv^2) for the observations in the period 1846 to 1938 with respect to the final orbits. Given also are the Σv^2 of the observations from 1960 to 1968, which were compared with the orbits but which were not used in the fitting process, as well as the Σv^2 of all of the observations 1846 to 1968. The best representation of the longitude observations at the modern epoch 1960 to 1968 by extrapolation of an orbit based on observations from 1846 to 1938 was used as the criterion in determining a new mass of Pluto. A parabola was fitted through these Σv^2 of the longitude for the reciprocal masses 930,000; 1,500,000; and 2,640,000. Differentiation of the parabola indicated a reciprocal mass of 1,812,000 as the value best representing the observations.

Table 2. Osculating elliptic elements representing the new numerical theory and the changes from the elements of Eckert, Brouwer, and Clemence (*8*). *M*, mean anomaly; ϖ, argument of perihelion; Ω, longitude of the ascending node; *i*, inclination; *e*, eccentricity; and *a*, semimajor axis (A.U.).

Elements	Values	Change
M	133° 44′ 9.783″	−2′ 49.353″
ϖ	270° 3′ 30.833″	+2′ 48.149″
Ω	131° 16′ 41.893″	− 2.598″
i	1° 46′ 33.651″	+ 0.244″
e	0.0118570458	+0.0000024223
a	29.9871290465	+0.0003408578

A final orbit obtained with the use of this mass of Pluto and fitted to the observations 1846 to 1938 represents the observations 1960 to 1968 (Fig. 3). The Σv^2 of the observations in orbital longitude and latitude determined from this mass of Pluto are indicated at the end of Table 1. An attempt to represent all of the observations from 1846 through 1968 by an adjustment of the Eckert, Brouwer, Clemence orbit resulted in Σv^2 in longitude of 37.76 arc sec and in latitude of 22.06 arc sec. It is evident that an orbit incorporating a reciprocal mass of Pluto of 360,000 cannot satisfy the observations in longitude over this arc as well as the new orbit with the reciprocal mass of 1,812,000 can. The slight increase in the Σv^2 in orbital latitude for the reciprocal mass 1,812,000 as compared to that for 360,000 is puzzling but not unexpected. Because of the large range of Pluto's latitude in the period 1846 to 1968, the latitude residuals of both Uranus and Neptune should be sensitive indicators of the mass of Pluto. However, Brouwer (*4*), in an analysis of observations of Uranus, has shown the presence of an unexplained systematic trend in the latitude residuals that cannot be removed by either an adjustment of the orbital elements or the mass of Pluto. The slight degradation in representation of the latitude observations of Neptune with the new mass of Pluto is therefore of doubtful significance.

If Pluto has the same density as Earth, then the new determination of its mass (0.18 Earth masses) indicates a diameter of 7200 km. On the other hand, if the upper limit of the diameter of Pluto (6400 km) is accepted, then Pluto must be at least 1.4 times as dense as Earth. The osculating elliptic elements (epoch Julian day 2430000.5) representing the new numerical theory are given in Table 2 with an indication of the changes from the elements of the theory of Eckert, Brouwer, and Clemence. Further refinement of the value of the mass of Pluto and the elements of the orbit of Neptune must await completion of a systematic discussion of the observations of Neptune now being made at the U.S. Naval Observatory.

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