transformation in animal cells. There are concrete indications, however, that in spite of their short half-life, proteins can act as carriers, as precursors of active agents, and as regulators of metabolic functions in host cells. They may also be important in the control of growth and differentiation. These functions of exogenous proteins are still largely unexplored.

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Are "Infinity Machines" Paradoxical?

Can processes involving an infinite sequence of operations or "acts" be completed in a finite time?

Adolf Grünbaum

Let us begin with a statement of the problem.

1) Consider a hypothetical machine, to be called the "π-machine," which prints *all* the digits 3.1415926535 . . . constituting the infinite decimal representation of π in such manner that the first digit is printed in ½ minute, the second in ¼ minute, the third in ⅛ minute, and so on. We disregard here the question of whether an infinite time might not be required for the more complicated process by which this progression of digits might first have been computed seriatim, via, say, Archimedes's method of exhaustion for determining the area of a unit circle. It suffices for our purposes that a progression of digits is to be printed as described, and, if necessary, these digits may be any digits whatever, provided that there is a denumerable infinity of them. Furthermore these $\aleph_0$ numerals might all have been inserted simultaneously into the printing press in a spatial arrangement discussed below. I refer here to any such process as "the π printing." And our problem is to determine the conditions, if any, under which the π printing could be completed in 1 minute.

2) Let a hypothetical mechanical device capable of reciting the sequence of natural numbers $n = 1, 2, 3, \ldots$ depart from the leftmost point 1 and move continuously to the right through a unit interval in 1 minute to the point 0. Now focus on the progression of points $1/n$ (where $n = 1, 2, 3, \ldots$) within that interval, a progression

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which contains the point 1 but not the point 0. And suppose that the device might perform recitations as follows: for every one of these points $1/n$ (where $n = 1, 2, 3, \ldots$), when reaching that point it begins to recite the number n and completes the recitation of n by the time it arrives at the next point in the progression. Thus, for every natural number n, the device takes $(1/n) - [1/(n+1)]$ of a minute to complete its recitation. It does not, however, use the English name sounds for the natural numbers; instead it uses a sequence of names whose successive lengths are governed by a restriction discussed below. I refer to this traveling number-reciting device as "the Peano machine," in honor of the man who axiomatized the theory of natural numbers. Our problem is this: Will the prescribed names of all the natural numbers have been recited when the Peano machine reaches the point 0 after 1 minute?

3) There are reading lamps equipped with buttons which, if pressed, switch the lamp on when it is off and switch it off when it is on. If the lamp is off and the button is pressed an odd number of times, the lamp will be on, but if the button is pressed an even number of times, the lamp will be off. Let the lamp be off, and now suppose that the first jab of the button requires ½ minute; the second, ¼ minute; and so on. Our problem is this: Under what conditions, if any, can such jabbing of the button switch the lamp on and off $\aleph_0$ times within the finite time of 1 minute? Since J. F. Thomson introduced the putative process of these $\aleph_0$ on-off lamp switchings into the literature (*1*), I refer to it as "the Thomson process."

4) A unit space interval includes the infinite geometric series of nonoverlapping subintervals

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots .$$

Any infinite progression of intervals which converge to zero geometrically will here be called a "Z-sequence." This designation is intended to honor the Greek philosopher Zeno of Elea. Zeno's celebrated paradoxes of about 2500 years ago have haunted modern mathematical kinematics. For one of the insistent questions posed by his so-called Dichotomy paradox was the following: How can the *completion* of a runner's traversal of the *unending* Z-sequence be *temporally intelligible*? To deal with the temporal intelligibility of the mathematical apparatus of modern kinematics, we shall consider the issue of completing each of two processes as follows:

a) The traversal of the Z-sequence in unit time by a runner who runs *continuously* at uniform unit velocity. This runner will traverse the first Z-interval in ½ unit of time, the second in ¼ unit of time, and so on ad infinitum. I refer to this process as the "legato Z-run."

b) The traversal of the Z-sequence in unit time by a runner who runs *discontinuously* as follows: he takes ¼ unit of time to traverse the first Z-interval of length ½ and rests for an equal amount of time; then he takes ⅛ unit of time to traverse the second Z-interval of length ¼ and rests for an equal amount of time; and so on ad infinitum. I refer to the latter process as the "staccato Z-run."

In regard to the legato motion, Zeno asks: How can the runner's completion of such a Z-run possibly be temporally intelligible if its completion requires the elapsing of a progression of temporal subintervals which is *endless* and whose durations are $1/2^n$ ($n = 1, 2, 3, \ldots$)? We can raise his question as well about our staccato Z-runner who is required to traverse each Z-interval in half the time required by the legato runner and then to wait for the latter to catch up with him before traversing the next Z-interval. Thus, if one imagines that the two runners depart simultaneously and run parallel to each other on essentially the same race course, then we can ask: Do they arrive simultaneously at their final destination after a finite time? These questions cannot nowadays all be dismissed as mere mathematical anachronisms, born of failure to understand the modern arithmetical theory of convergent infinite series. For they have been asked by eminent contemporary mathematical physicists. Thus, referring to "Zeno's well-known paradox of the race between Achilles and the tortoise," Hermann Weyl writes (*2*, pp. 41–42):

> The remark that the successive partial sums
>
> $$1 - \frac{1}{2^n}\ (n = 1, 2, 3, \ldots)$$
>
> of the series
>
> $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$
>
> do not increase beyond all bounds but converge to 1, by which one nowadays thinks to dispose of the paradox, is certainly relevant and elucidating. Yet, if the segment of length 1 really consists of infinitely many subsegments of lengths ½, ¼, ⅛, . . . , as of "chopped-off" wholes, then it is incompatible with the character of the infinite as the "incompletable" that Achilles should have been able to traverse them all. If one admits this possibility, then there is no reason why a machine should not be capable of completing an infinite sequence of distinct acts of decision within a finite amount of time; say, by supplying the first result after ½ minute, the second after another ¼ minute, the third ⅛ minute later than the second, etc. In this way it would be possible, provided the receptive power of the brain would function similarly, to achieve a traversal of all natural numbers and thereby a sure yes-or-no decision regarding any existential question about natural numbers!

5) Modern kinematical theories assume that between any two instants of time there is at least one other, and that between any two points of space there is at least one other. Any class of elements ordered by this kind of betweenness is said to be "dense." It is thus logically possible that diverse classes of elements are each dense with respect to their particular ordering relations of betweenness. And in the case of any one class of elements and a specified betweenness relation, we can ask, therefore, whether the class constitutes a dense system.

For any given instant of a dense temporal order, there does not exist any immediately *next* instant either before it or after it, since *any* two instants are separated by others. Thus for any two instants t_1 and t_2, there there are infinitely many other instants between them. This dense kind of betweenness is to be contrasted with the betweenness associated with a discrete series. In a discrete series like 1, 2, 3, . . ., the elements are ordered consecutively—that is, with one immediately next to the other, which is either an immediate successor or an immediate predecessor. And if there are any elements between two others in a discrete series, their interposition is consecutive. Newtonian physics, relativity theory, and standard quantum mechanics all assume that spatial and temporal betweenness are both dense as opposed to discrete. Standard quantum theory has "discretized" or quantized some physical properties whose counterparts in classical physics were mathematically continuous. But standard quantum theory has *not* quantized space or time, and the time of that theory is dense. For in standard quantum theory, every point of continuous physical space is a potential sharply defined position of, say, an electron, and, separately, every instant of a continuous time is potentially the time of a physical event. And this assertion of the continuity of space and time fully allows that theory to repudiate the well-defined particle trajectories of Newtonian or relativistic mechanics.

By assuming that physical time and space are each a linear mathematical continuum of elements, the modern kinematical theories assert a time or space interval to be an actually infinite dense set of punctal elements. Thus the spatial and temporal intervals (continua) of modern kinematics are *actually* infinite sets of punctal elements in the sense of Georg Cantor, not merely *potentially* infinite aggregates in Aristotle's sense; Aristotle's aggregates are infinite merely in the sense that their members are infinitely divisible. Hence, in modern kinematics the members of an infinite set of nonoverlapping subintervals of an interval of time or space cannot be regarded as first having to be generated by hypothetical division operations on the total interval.

In the wake of Zeno, such scientist-philosophers as A. N. Whitehead and William James denied that the temporal order can intelligibly be dense. Inviting attention to the perceived relations of temporal order among events *as they are actually happening*, they claimed that this order is discrete and exhibits nextness or consecutivity as opposed to denseness. We are told that, ordinally, time as perceptually experienced by us is the order of the actual coming into being of events, and thus the order of the successive nows of awareness. Note, we are told, the content of our perceptual awareness of a runner's motion as it is taking place, and attend to the temporal order in which it is perceived as actually happening. Then our actual experience of its happening —as distinct from our retrospective awareness of the entire motion—has the following feature: there is a first event of the motion, constituted by the runner's presence at his point of departure; a temporally *next* event right after the departure event; consecutively ordered temporally intermediate events; a temporally *next*-to-the-last event right before the runner arrives at his destination; and the terminal event of the motion, constituted by his arrival there. Clearly, we are told, the nextness of before and after characteristic of actual happening is logically incompatible with a dense temporal order. For in what sense, asks James, can the events of the motion intelligibly be said to succeed one another *temporally* if the initial instant of the motion is not followed by any immediately later *next* instant because a dense infinity of instants must first occur between that initial instant and *any* later one, however close? And he replied that the actual elapsing of ongoing time cannot intelligibly be dense. Hence James and Whitehead contend that, if there is to be a serial order of nonsimultaneous events which is temporal at all, that order of actual happening must be consecutive in the manner of a discrete sequence.

Our problem is, therefore, to see whether a dense temporal order of punctal physical events is indeed paradoxical, as James and Whitehead claim it is. When dealing with this ordinal question, we shall not be concerned with the following *metrical* challenge inherent in Zeno's different paradox of extension: How can chronometry and geometry devise physically reasonable rules for adding durations and lengths which would allow an extended interval to consist of *unextended* instants or points? Specifically, Zeno rejected, as physically paradoxical, additivity rules for length and duration which permit kinematical theory to assert *each* of the following two assumptions in a formally consistent way: (i) a line segment of physical space, whose length is positive, is a linear mathematical continuum of points, each of which is of length zero; (ii) the time interval corresponding to a physical process of positive duration is a linear mathematical continuum of instants, each of which is of zero duration. Zeno's allegation of such a paradox of extension was endorsed by the contemporary Nobel-laureate physicist P. W. Bridgman, who wrote (*3*):

> With regard to the paradoxes of Zeno . . . if I literally thought of a line as consisting of an assemblage of points of zero length and of an interval of time as the sum of moments without duration, paradox would then present itself.

Limitations of space prevent my presenting here a vindication of kinematics vis-à-vis this particular charge of paradox. I must therefore refer the reader to the detailed statement given in my recent book on the topic of this article (*4*).

The various allegations of paradox made concerning the "infinity machines" and the runners mentioned above pertain to the distinctively kinematical aspects of the specified processes. Hence their chemical, electrical, physiological, or other feasibility is not at issue, unless it has a bearing on the assessment of their possibility from the standpoint of kinematics.

I discuss the legato motion first. Here it is taken for granted that the runner can indeed traverse a unit space interval in unit time, as allowed by kinematic theory. But the allegation of paradox is the charge that, by affirming a dense temporal order, this very theory also permits the deduction that the runner cannot reach his destination in a finite time. In this way, kinematic theory is charged with entailing the impossibility of the processes which it purports to describe.

The Legato Motion

Zeno calls attention to the fact that, if the runner is to traverse a unit space interval in unit time, he must, among other things, successively traverse in corresponding times the progression of nonoverlapping spatial subintervals whose lengths are given by the numbers

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots \quad (n = 1, 2, 3, \ldots).$$

How then, Zeno asks, can this process possibly be completed by the runner in a finite time, if its completion requires the elapsing of a progression of temporal subintervals which is *endless* as a consequence of the denseness postulate, and whose durations are $1/2^n$ ($n = 1, 2, 3, \ldots$)?

It will be useful to adopt the terminology used by Vlastos (*5*) and to refer to the traversal of any of the spatial subintervals of our Z-sequence as "making a Z-run." Vlastos notes that the term *run*, as commonly used, individuates uniquely the physical action to which it applies, much as *heartbeat* does. And he points out that in this sense of "run," the runner's traversal of the Z-sequence could only be described as a *single* run and not as having involved a denumerable infinity of Z-runs. But clearly, to traverse the unit interval in one smooth and uninterrupted "run" in the ordinary sense, the runner must—among other things —traverse all the members of the Z-sequence and, in the latter sense, make $\aleph_0$ Z-"runs." To distinguish between these two quite different uses of the noun *run*, Vlastos writes "run_a" for the single motion which we can perceive with our unaided senses in daily life and "run_b" for the kind relevant to the Z-sequence of kinematics.

Human awareness of time exhibits a positive threshold or minimum. This fact can now be seen to have a consequence of fundamental relevance to the appraisal of Zeno's argument. For it entails the conclusion that none of the infinitely many temporal subintervals in the progression whose magnitude is less than the human minimum *perceptibilium* can be experienced as elapsing in a way that does metrical justice to its actual duration. To succeed, the attempted individual contemplation of all the subintervals would require a denumerable infinity of mental acts, each of which requires or exceeds a positive minimum duration. Instead of experiencing these subintervals as elapsing in a metrically faithful way, we gain our metrical impression of duration in this context from the time needed by our mental acts of contemplation and not from the respective duration-numbers which we associate intellectually with the contemplated subintervals when performing these mental acts! And the resulting compelling feeling that an infinite time is actually needed to accomplish the traversal in turn gives rise to the impression that this paradoxical result is deducible from the theory of motion. Specifically, the existence of a duration-threshold of time awareness guarantees that there is a positive lower bound on the duration of any run_a. And this fact enters into several of the following fallacies of Zeno's Dichotomy paradox.

1) Zeno's claim that the progression of Z-run_b's requires an infinite future time is made plausible by a tacit appeal to our awareness that $\aleph_0$ run_a's would indeed last forever, because there is a positive lower bound on the duration of any run_a. The threshold governing our acts of awareness likewise compels the feeling that, after the first instant, a unique next event must occur in the motion *and* that there must be a unique next-to-the-last event that occurs before the final instant of the motion, if there is to be a final instant at all. But the one-by-one contemplation which Zeno invites is not metrically faithful to the actual physical durations of the contemplated subintervals, which converge to zero by decreasing geometrically. And Zeno illicitly trades on the fact that our intuitive time awareness rightly boggles at *experiencing* each of $\aleph_0$ subintervals of time as elapsing individually. But, justified though it is, this boggling cannot detract from the fact that any and every temporal subinterval of the motion is over by the end of one unit of time: for every n, the sum S_n of the first n terms of the geometric series of duration numbers

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}$$

is given by

$$S_n = 1 - \left(\frac{1}{2}\right)^n \quad (n = 1, 2, 3, \ldots),$$

which is less than 1. It follows that both distributively and *collectively*, all $\aleph_0$ temporal subintervals of the motion elapse within one unit of time. The justification for this conclusion becomes further apparent when one becomes cognizant of the next error, by which Zeno buttresses his conclusion that the runner would *never* reach his destination.

2) With respect to the relation of temporal precedence, the set comprising the temporal subintervals of the progression *and* the instant of the runner's arrival at his point of destination has the form of an infinite progression followed by a last element and is said to be of ordinal type $\omega + 1$. Furthermore, the instant of the runner's arrival at his destination point 1 does not belong to any of the subintervals of the progression. Thus, the closed time interval required by the runner's total motion consists of all the instants belonging to any of the subintervals of the progression *and* of the instant of arrival at point 1. By failing to include the instant of arrival, the membership of the subintervals of the progression fails to exhaust the entire membership of the closed time interval required by the complete motion.

Zeno illicitly exploits the fact that it is logically impossible to find the terminal instant of the motion in any of the subintervals of the unending progression. For he appeals to this fact as lending further credence to his claim that the union of the subintervals of the progression is of infinite duration. But the logical impossibility of finding the terminal instant in any of the subintervals forming the unending progression means no more than that this instant is not to be found in a time interval from which it has been excluded and which has been left half open by its exclusion; the half-openness of the resulting time interval does not show that the union of the subintervals must be of infinite duration just because it has no terminal instant and just because the infinite progression of subintervals has no last member. For the terminal instant is the *earliest* instant following every instant belonging to any subinterval of the unending progression, while the durations of these subintervals suitably converge to zero. The nonexistence in the progression of a last subinterval during which the motion would be completed does not preclude the existence of an instant later than all the subintervals which is the last instant of the motion.

This state of affairs is expressed arithmetically in the following way:

1) If the runner departs at $t = 0$, then, corresponding to the nonexistence of a last temporal subinterval of the motion in the progression, the respective times by which he has traversed the successive Z-intervals are given by the infinite sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots, \frac{2^n - 1}{2^n}, \ldots \quad (n = 1, 2, 3, \ldots).$$

2) Although the number 1 is not a member of this infinite sequence of time numbers, the arithmetic limit of this infinite sequence on the number axis is constituted by the number 1, which is the time coordinate of the last instant of the motion and represents the total duration of the union of the subintervals belonging to the progression.

3) The runner traverses ever shorter subintervals of the unit race course in proportionately ever shorter subintervals of time, thereby traveling at constant speed.

What then are we to think of the charge that the arithmetic theory of limits has been lifted uncritically out of the context of its legitimate application to physical *space* and adduced irrelevantly in an effort to refute Zeno's allegations of *temporal* paradox? The considerations of this section show that the mathematical apparatus of the theory of limits is ordinally and metrically no less appropriate to physical time than it is to physical space. Note that I have not invoked the *arithmetic* theory of limits, as such, to dismiss the allegation that kinematical theory entails temporal paradoxes. Instead, my contention has been that the ordinal and metrical structure of physical time provides justification for applying that arithmetical theory, and that Zeno's specific deductions of metrical contradictions in the Dichotomy are each vitiated by fallacies which I am engaged in pointing out.

The highly misleading role played by Zeno's one-by-one contemplation of the members of his progression becomes conspicuous when one notes the following facts. It would even take us forever to contemplate, one by one, the progression of durationless instants which divide one temporal subinterval from the next, and yet the durational measure of this progression of instants is zero! By the same token, the fact that our contemplation of the $\aleph_0$ subintervals would last forever is not a basis for concluding that the union of the progression of these subintervals would be of infinite duration. In summary, Zeno would have us infer that the runner can *never* reach his destination just because (i) we could never contemplate, one by one, all the subintervals of the progression, for want of time, and (ii), for purely logical reasons, we could never find the terminal instant of the motion in any subinterval of which it is not a member, and it is not a member of any of the subintervals of the progression. But it is altogether fallacious to infer Zeno's conclusion from these two premises.

The recent literature on Zeno continues to provide illustrations of the intellectual havoc that has resulted from an irrelevant, though tacit, appeal to the fact that there is a positive lower bound on the duration of any single mental act of ours, such as conscious counting. Thus, G. J. Whitrow (*6*) seems to have engaged in precisely such an unwitting appeal in his endeavor to show that the denseness which we attribute to finite intervals of space cannot similarly be attributed to physical time without thereby generating logical antinomies. After stating (*6*, p. 148) that "we must not assume that . . . in *time*, any infinite sequence of operations can be performed," Whitrow considers the consequences of assuming that the runner (that is, Achilles) passes through the entire progression of positions envisaged by Zeno as the respective termini of his subintervals. Whitrow invites us to assume that, in so doing, the runner would number all these positions consecutively, and he concludes (*6*, p. 148) that then the runner's task would involve exhausting "the infinite set of positive integers by *counting*." In this way Whitrow misidentifies the metrical features of the process of (conscious) counting in English with those of traversing Zeno's progression of points in a finite interval, in a manner akin to Zeno's illicit appeal to the eternity of one-by-one contemplation in the Dichotomy.

The Staccato Motion

The staccato runner at no time lags behind his legato colleague during the closed unit interval but is either ahead of him or abreast of him. While running within each of the Z-intervals, the staccato runner's average velocity is twice that of his legato colleague, but his overall average velocity for the total interval is equal to his colleague's velocity and is less than the velocity of light in a vacuum. It follows that if the legato runner reaches his destination in 1 unit of time after traversing the Z-sequence, then so also does the staccato runner. And this conclusion has the following important consequence: Given that the pauses separating the individual traversals made by the staccato runner form a geometric progression whose terms converge to zero, it is immaterial to the traversability of the total unit interval in a finite time that the process of traversal consists of $\aleph_0$ motions separated by pauses of rest (staccato run) instead of being one uninterrupted motion which can be *analyzed* into an infinite number of submotions (legato run). And if we wish to call the staccato runner's execution of the $\aleph_0$ separate motions "doing infinitely many things," then his performance shows that infinitely many things can be done in a finite time (*7*). What could reasonably be expected here in the way of a "proof" that the staccato run is kinematically possible is the following: A demonstration that, given the kinematical principles of the theory and the boundary conditions, the theory entails the *finitude* of the total duration of the staccato run. For the allegation that that run was impossible was based on its allegedly infinite duration. Thus, I am engaged in giving a proof of the physical possibility of the staccato run. Of course, if the pauses between the individual traversals of the staccato run were all equal, then this run could not be carried out in a finite time, no matter how small each of the equal pauses might be. But the introduction of the assumption of *equal* pauses is clearly contrary to the stated condition of geometrically decreasing pauses. No wonder, therefore, that a surreptitious appeal to the assumption of equal pauses insinuates the actual deducibility of a paradox from the theory.

To obviate some objections to my contention that it is physically possible kinematically for the staccato run to be consummated in a finite time, I call attention to several points, as follows.

1) I was careful *not* to require my staccato runner to plant a flag at each of his Z-stops during the times when he suspends his motion to permit his legato colleague to catch up with him. For the erection of a flag at each of the $\aleph_0$ Z-stops would presumably require him to translate his own limbs and rotate the flag each time through a minimum positive distance, however small. And in that case the staccato runner would have to perform $\aleph_0$ equal spatial displacements in a finite time and thereby effect a spatially infinite total displacement of his own limbs and of the flags, in the following manner. The successive vertical velocities of his limbs required to plant the flags consecutively would increase boundlessly (though not monotonically!) with time up to the instant at which he comes to rest at his destination. But such a motion has two kinematically objectionable features: (i) at the instant $t = 1$ of arrival at the destination point P, the motion violates the requirement that the *position* of a body be a continuous function of the time, since the vertical position does not approach any limit as $t \rightarrow 1$, and, a fortiori, the vertical position does not approach the point P as a limit as $t \rightarrow 1$; (ii) the fluctuating *velocity* function has an instant of *infinite discontinuity* (at the time $t = 1$), since the plot of the runner's vertical position against time is a function of unbounded variation in the time interval (0, 1) and the velocity function is unbounded in every neighborhood of the terminal instant $t = 1$ (*8*).

By contrast, at $t = 1$ the staccato runner's horizontal position is a continuous function of time, and his horizontal velocities fluctuate only between two fixed values 0 and k. His horizontal *accelerations* do increase and decrease boundlessly as $t \rightarrow 1$, in the sense that the same velocity change takes place in ever shorter times. But this can reasonably be regarded as kinematically innocuous. For the horizontal *velocity* function has an instant of only finite discontinuity at $t = 1$ (just as the graph of a step function has points of only finite discontinuity), while the horizontal *position* is indeed a continuous function of the time. That there are physically reasonable cases in which the position x of a particle varies continuously with time t while either the velocity or the acceleration does not, emerges from the following different case of temporally "semi-parabolic" motion along the x-axis: $x = t^2$ for $t \geqslant 0$, and $x = -t^2$ for $t < 0$. Here both the position and the velocity functions are continuous at $t = 0$, but the acceleration changes discontinuously from -2 to $+2$ at that instant.

It might be asked why I gave kinematic sanction to an infinite discontinuity in the horizontal acceleration at $t = 1$ after having objected to the infinite discontinuity in the vertical velocities required to plant the flags. To this I reply: The former is here associated with a *continuous* time-dependence of the horizontal position, whereas the latter is a consequence of a *discontinuous* change in the vertical position. Thus, it may be that a given kind of discontinuity is kinematically permissible in the case of a higher-order derivative (for example, acceleration) but not in the case of the corresponding lower-order derivative (for example, velocity).

2) In view of the thresholds which govern the physiological reaction times of the staccato runner and his times of conscious execution of a set of instructions, it is clear that this runner cannot be "programmed" to perform the staccato run in accord with the required metrical specifications when the Z-intervals and the corresponding times of traversal become small enough to fall below his thresholds. But this fact does not vitiate my contention that, in principle, kinematically the staccato run, as described, is physically possible. For kinematic theory allows us to assume that the staccato runner's separate motions have the prescribed metrical properties.

3) There *may*, of course, be specifically dynamical—as distinct from kinematical—difficulties in effecting the infinitude of horizontal accelerations and decelerations required by the runner's alternate starting and stopping. Thus, if we oversimplify the Newtonian treatment and consider the succession of accelerations of the body of fixed mass m from rest to a velocity of two units, then Newton's Second Law, $F = m\ (dv/dt)$, tells us that

$$F \cdot \Delta t = m \cdot \Delta v.$$

But the successive times Δt_n ($n = 1, 2, 3, \ldots$) available for imparting the same velocity change Δv to the constant mass m converge to zero as the Z-intervals decrease. Hence the successive force values F_n have to become proportionately indefinitely large to assure the constancy of the product $F_n \Delta t_n$. And if we assume that the successive forces are given by $F_n = m\ a_n$ and act through distances Δx_n given by $\Delta x_n = (1/2)\ a_n \Delta t_n^2$, then we have

$$F_n \Delta x_n = \frac{1}{2} F_n a_n \Delta t_n^2 = \frac{1}{2m} (F_n \Delta t_n)^2.$$

But we saw that all the nonzero products $F_n \Delta t_n$ are equal. Hence each of the $\aleph_0$ products $F_n \Delta x_n$ has the same nonzero value. But this means that the total energy (work) expended by the runner in imparting the same finite velocity change Δv to his body $\aleph_0$ times is infinite. Thus, the runner would have required an infinite store of energy when he set out on his run. For he sustains $\aleph_0$ *uncompensated* losses of kinetic energy in the decelerations, and the total magnitude of these losses is infinite.

Let us disregard these specifically dynamical difficulties but be mindful of ruling out any flag-planting or other marking processes that would require any discontinuous change in any component of the displacement of the runner's limbs. I believe I have shown that, in the absence of such changes, kinematically it is physically possible for the staccato runner to reach his destination as prescribed in a finite (unit) time. Indeed he can be held to have "marked" each end point of the progression of end points of the Z-intervals by the act of stopping at each one for the prescribed length of time while awaiting his legato colleague. If this waiting at the Z-stops may be considered a "marking" procedure, then my staccato runner's total motion constitutes an important counterexample to one of the theses recently put forward by C. S. Chihara as part of his interesting critical response to Weyl's comparison of the Z-run with the performance of an infinity machine. Chihara believes that, for logical reasons, the difference between Achilles's mere traversal of the interval and Achilles's marking of all the end points of the Z-intervals in the course of his journey makes for the difference between the possibility of completing the run in a finite time and the requirement for an infinite time. He says (*9*):

> . . . to give a more intuitive characterization of the difference between Achilles' journey and Achilles' task of marking the end points, in the former case we start with the task and analyze it into an infinite sequence of stages, whereas in the latter case we start with the stages and define the task as that of completing the infinite sequence of stages. To complete the journey, one must simply perform a task which can be analyzed ad infinitum, but to complete the task of marking all the end points, one must really do an infinite number of things.

But as we saw, the staccato runner does "really do an infinite number of things" in what is kinematically a demonstrably finite time. It would appear that here Chihara is mistaken as to the source of the difference between the possibility of completion in a finite time and the requirement for an infinite time. The staccato run is *not* one uninterrupted motion which can be merely analyzed into an infinite number of submotions, as in the case of the legato run. Instead it consists of $\aleph_0$ motions separated by pauses, and yet kinematically its completion in a finite time is physically possible under the stated conditions.

The Pi-Machine

In considering kinematically whether it is physically possible for the π-machine to accomplish the π-printing in a finite time, I must immediately stipulate that the heights from which the press descends to the paper to print the successive digits *cannot* be equal but must form a geometrically decreasing series converging to zero. In this way I can ensure that the spatial magnitude of the successive tasks does not remain the same while the time available for performing them decreases toward zero. Like the runners, the π-machine is thereby called upon to move at only a constant average speed by traversing ever smaller distances in proportionately ever smaller times. My reason for requiring the heights from which the press descends to converge to zero in a suitable fashion is apparent from the analysis given above of the flag-planting in the case of the staccato runner. If these heights did not converge to zero, the successive velocities required for the printing would soon exceed the velocity of light and would vary with time in a manner that is kinematically objectionable even in the context of the Newtonian theory. Here, no less than in the case of the staccato runner, I ignore the dynamical problems of programming the π-machine so that the successive spatially and temporally shorter descents of the press can be triggered as required.

I require, furthermore, that the widths of the successive numerals to be printed converge to zero in such a way that *all* the $\aleph_0$ digits can be printed in a horizontal line on a *finite* strip of paper. In laying down this second requirement, I blithely ignore as kinematically irrelevant the blurring of the digits on the paper through smudging of the ink when their widths become sufficiently small, not to speak of the need for ink droplets smaller in diameter than an electron!

Under the fundamental restriction of my first proviso, regarding the heights of descent, the π-printing no more requires an infinite time than the legato or the staccato runner does. And, given my second requirement, concerning the widths of the successive digits, the spatial array of the $\aleph_0$ digits no more requires an infinite space than does the unending progression of Z-intervals which collectively fit into the space of a finite unit interval. As long as the sequence 3.1415926535 . . . is printed so that the successive widths of the digits converge to zero in the manner of the Z-intervals, the question "What does this array look like at the right-hand end?" receives the same kind of answer that the corresponding question about the progression of Z-intervals receives. We must not make a misguided attempt to form a visual picture of the open end of a finite, half-open space interval, and we are aware that the metrically finite union of the Z-intervals is open at the right "end," as is the total space interval formed by the progression of horizontally shrinking digits. Although we cannot visually picture the nonexistence of a rightmost point, our very characterization of the openness of the right end shows that we clearly understand in ordinal terms "what that end looks like." Just as the interval constituted by the union of the Z-intervals can be closed at the right end by the addition of a rightmost (last) point, so also, of course, can the interval formed by the horizontal cross section of the unending π-sequence.

Precisely analogous remarks apply to the time intervals that correspond to (i) the process of traversing all the Z-intervals, and (ii) the process of printing all the digits of π as specified. In the case of Zeno's Z-runner, we naturally tend to include in the motion process the event of his arrival at his destination, where he *first* comes to rest: in so doing, we seem to be interested not only in those states of the legato runner in which his velocity is positive but also in the earliest of his states of rest. But in the case of the π-printing process, there may be a tendency to include in the printing process only those states of the π-machine belonging to the printing motion and to exclude the earliest subsequent event which is not involved in the printing. Thus, by virtue of our decision on whether or not to include a terminal event in a given temporally finite process, the time interval corresponding to either of the Z-motion processes turns out to be closed at the later end, whereas the time interval corresponding to the π-printing process does not. But the exercise of our option to omit from the time interval corresponding to the π-printing the *earliest* instant following all the instants during which the press is busy printing must not lead us to draw the fallacious inference that no such earliest subsequent instant exists and that the π-printing cannot be completed within a finite time after its start. One might as well infer that the spatial interval constituted by the union of the unending progression of Z-intervals must be spatially infinite!

Let me assume that I am right in claiming that the completion of the kind of π-printing process which I have described is physically possible kinematically, no less than the completion of the total staccato Z-run. Then my π-printing process constitutes a further counterexample to Chihara's claim, noted above, that, purely kinematically, an infinite number of things cannot be done in a finite time.

Suppose that I had not explicitly ruled out the equality of all the heights from which the printing press is to descend but had countenanced their equality. In that case, we can conclude the following from our discussion of the staccato runner. Quite apart from the fact that the speeds greater than the speed of light required by this equality would be incompatible with the special theory of relativity, these required speeds would be sufficient to ensure the kinematic impossibility, according to Newtonian physics, of the completion of the printing in a finite time. For the required unbounded speeds would not accord with the demand that the velocity function of a body may not have an instant of infinite discontinuity, as we noted apropos of the flag-planting. Thus, unless Weyl can show that the successive spatial displacements (or "tasks") performed by a machine in calculating (not just printing!) seriatim the digits of π can, in principle, suitably converge to zero, my account of the π-machine does sustain the following conclusion reached by Chihara (*9*): Weyl was mistaken in claiming that only if an infinite sequence of calculations can be completed in a finite time can Achilles traverse *all* the Z-intervals.

The Peano Machine

If we were to allow the use of the English-language names of the numbers to be recited by the Peano machine, then there would be a number beyond which the lengths of the names —each measured by the name's syllable content—would increase boundlessly. And even if these name lengths remained the same, the "syllable-size" of the successive recitation tasks would remain the same while the time available for their performance would decrease indefinitely. To assure that the average speed of the mechanical "lips" engaged in the recitation could remain constant instead of having to increase boundlessly, we would require *non*-English names of the successive numbers such that the successive distances traversed by the mechanical lips as they perform their recitations would decrease in proportion to the available time. It is quite unclear how distinctive names capable of being pronounced by the mechanical lips in accord with this stringent requirement could be generated by a rule.

Let us postpone this difficulty for the moment and turn from the modulating mechanical lips to the vibrating membrane of the mechanical voice. We note that each of the required $\aleph_0$ distinct sound-names or noises requires at least one vibration of the voice membrane. But the time available for the utterance of these successive noises converges to zero. Hence the *frequency* of the noises and also of the vibration of the membrane must increase indefinitely. It has been suggested to me by A. Janis that the ensuing denumerable infinity of frequencies permits each natural number to be named by a sound of distinctive pitch. And it seems to me that such a pattern of noises constitutes an acceptable code language for numbers.

The energy imparted to the air particles by the vibrator is proportional to the square of the frequency and to the square of the amplitude. We can ensure, though, that the frequency pattern required by the total recitation does *not* necessitate the expenditure of an infinite amount of energy in a finite time. For although the frequencies of the membrane must increase indefinitely, we can require that the amplitudes of the successive vibrations of the total recitation decrease in such a way that the total energy expended is finite.

Not only on these dynamical grounds is a decrease in the successive amplitudes important. For in order that the vibratory motion of the membrane be kinematically possible, the amplitudes of the vibrations corresponding to the successive noises must decrease and suitably converge to zero. Even if the membrane executed only one vibration for each noise, it would have to vibrate through an infinite total distance in a finite time if the amplitudes of all the $\aleph_0$ noises were equal.

The assumed fulfillment of this proviso regarding the decrease in the amplitude does enable us to conclude that it is physically possible kinematically for the traveling Peano machine to complete the recitation in a finite time. Just as in the case of the π-machine, there is a tendency to think of the number-recitation process as not including the earliest of the Peano-machine states in which the machine is no longer engaged in reciting. This essentially classificatory decision on our part thus prevents the finite time interval required for the total recitation from being closed at its later end. But I remind the reader of the caveat I issued on this point apropos of the π-machine: It should not be inferred that a time interval *must* be metrically infinite just because it is ordinally open at either the later or the earlier "end."

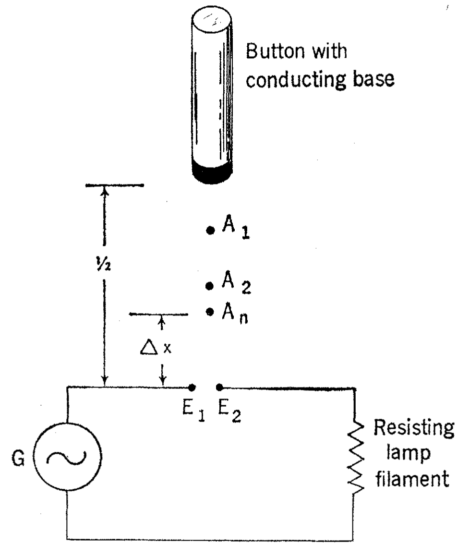

Fig. 1. Schematic diagram of the circuitry of the Thomson lamp.

The conditions which we had to impose to assure that the Peano machine can accomplish its $\aleph_0$ recitations in a finite time make further apparent why it was necessary to object, in the case of the legato runner, to Whitrow's linking of Achilles's traversal of a progression of points in a finite time with the expectation that Achilles would number them all by counting. I take it that the counting Whitrow had in mind would take the form of reciting all the natural numbers *in English*, or of performing the infinitude of threshold-governed mental acts of thinking of all of them seriatim. And we saw that either of these forms of counting would take forever. If we were to grant that Achilles can traverse the progression of points in a unit space interval *only* if he can thus count them all, then indeed Whitrow would be warranted in concluding that Achilles cannot accomplish the traversal in a finite time. But the separate temporal analyses which I gave of the processes of traversal and of vocal or mental counting in English show that Whitrow has misidentified the durational features of counting with those of Achilles's traversal of the progression of Z-points.

The Thomson Lamp

Let t_0 be the initial instant $t = 0$, and t_1 the terminal instant $t = 1$ at which the denumerable infinity of on-off switchings are to have been completed.

Let us simplify our consideration of the kinematic feasibility of this process by disregarding the question of the following electromagnetic possibility: the realizability of the conditions required for the emission of visible photons from the filament of the lamp bulb during each of $\aleph_0$ geometrically decreasing "on"-periods prior to t_1, and possibly at t_1 and thereafter. Instead, let any state in which the lamp circuit is merely electrically closed qualify as an "on"-state of the lamp, while an "off"-state is one in which the circuit is thus broken or open. We can then confine our consideration to the kinematics of the motions of the button or switch whose alternating states correspond to closed or open states of the circuit by virtue of the electrical coupling or decoupling between them.

Let a button be equipped with an electrically conducting base which can close a circuit when fitted into the space between the exposed circuit elements E_1 and E_2. And let the button be depressed through a *fixed* distance d to close the circuit every time the lamp is to be turned on, and pushed upward back to the starting position to break the circuit each of the $\aleph_0$ times the lamp is to be turned off. As is clear from the earlier discussion of the "infinity machines," if the button is to be at rest at time t_1 or to be moving at that time with a particular finite velocity, even according to Newtonian principles this arrangement would involve a kinematically impossible infinite discontinuity in the time variation of the button's velocity, quite apart from requiring relativistically prohibited velocities greater than the velocity of light.

There are certain conditions which must be satisfied by the switch (button) and the circuit elements to make Thomson's process, as specified by him, kinematically possible. And, as A. Janis has pointed out to me, these conditions are such that the state of the circuit at time t_1 is predictably closed. To see this, let us first recall the earlier discussion of the infinity machines. From this it is clear that the consecutive downward and upward jabs at the switching button, which alternately close and break the circuit, must produce displacements of the button whose lengths Δx are a suitably decreasing sequence converging to zero. And we must assume that there is no electrical arcing or sparking across *any* space gap Δx, however small, between the conducting button-base, on the one hand, and the exposed circuit ends E_1 and E_2 on the other. For if there were electrical arcing for all Δx equal to or less than some minimum ϵ, then the kinematic requirement that Δx suitably converge to zero as $t \rightarrow t_1$ would have the following result: there would be a time t_ϵ before t_1 such that the circuit would be electrically closed for all instants t belonging to the interval $t_\epsilon \leqslant t < t_1$. And this result would obviously violate Thomson's requirement that the lamp is still to be switched off $\aleph_0$ times during this time interval.

At time t_0, when the lamp is off, let ½ be the initial vertical distance between the button base and the horizontal circuit-opening E_1E_2. Then after the button has been depressed once from its initial position (with the base of the button at A) (Fig. 1) to close the circuit, let it be raised after each such depression *not* all the way to its initial position A but to intermediate points $A_1, A_2, A_3, \ldots, A_n, \ldots$ whose respective distances Δx from E_1E_2 are

$$\frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \ldots, \frac{1}{2^{2n+1}}, \ldots \quad (n = 1, 2, 3, \ldots).$$

Then the $\aleph_0$ circuit-closing jabs involve a sequence of downward displacements Δx

$$\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots, \frac{1}{2^{2n+1}}, \ldots \quad (n = 0, 1, 2, 3, \ldots).$$

The corresponding sequence of available time intervals Δt is

$$\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots, \frac{1}{2^{2n+1}}, \ldots \quad (n = 0, 1, 2, 3, \ldots).$$

If all of the downward motions were to proceed at unit velocity, then the circuit would be closed for only an instant each time during these particular time intervals. On the other hand, if only some fixed proper fraction $1/k$ of these available times Δt were devoted to the downward motions, then the velocity of the button would have the same value k each time, thereby satisfying the Newtonian kinematic requirement of an upper bound during the time interval $t_0 \leqslant t < t_1$. And, if the downward motion were to start each time at the *beginning* of the time interval available for it, the lamp would be on at least for the sequence of time intervals

$$\Delta t\,[1 - (1/k)]$$

—that is, at least during the time intervals

$$\frac{1}{2^{2n+1}} \left(1 - \frac{1}{k}\right) \quad (n = 0, 1, 2, 3, \ldots).$$

To conform to the requirements of the theory of relativity, the velocity k must be less than that of light in the units we are using. But even according to Newtonian principles the button velocity would impermissibly increase boundlessly, if only decreasing fractions $1/n$ ($n = 1, 2, 3, \ldots$), rather than a fixed proper fraction $1/k$, of the decreasing time intervals Δt were granted successively for the downward motions in order to secure successive on-states of durations

$$\frac{1}{2^{2n+1}} \left(1 - \frac{1}{n+1}\right) (n = 0, 1, 2, 3, \ldots).$$

Turning to the $\aleph_0$ circuit-breaking jabs, we note that they involve a sequence of decreasing upward displacements

$$\frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \ldots, \frac{1}{2^{2n+1}}, \ldots \quad (n = 1, 2, 3, \ldots).$$

The corresponding sequence of decreasing time intervals Δt available for breaking the circuit by upward movement of the button is

$$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots, \frac{1}{2^{2n}}, \ldots \quad (n = 1, 2, 3, \ldots).$$

Let $1/k$ be the particular fixed fraction of the available time interval Δt which is devoted each time to the button's circuit-breaking motion. Clearly $1/k \leqslant 1$, and the button's upward velocity v is given by $v = k/2$. Under the relativistic restriction that v have values less than c, the velocity of light, we also have $k < 2c$, or $(1/k) > (1/2c)$, so that

$$\frac{1}{2c} < \frac{1}{k} \leqslant 1.$$

Suppose that $1/k$ has some value in this interval other than 1, and let the upward motion terminate each time at the *end* of the time interval available for it. Then the lamp will also be on during the following initial positive subintervals of the intervals available for the button's upward circuit-breaking motions

$$\left(1 - \frac{1}{k}\right)\frac{1}{2^{2n+1}} \quad (n = 1, 2, 3, \ldots).$$

The state variable characterizing the lamp as either on or off is clearly a discrete variable, since it ranges over only two values rather than over a continuum of values. But we took a closed state of the circuit to be tantamount to an on-state of the lamp, while a broken state of the circuit is equivalent to an off-state of the lamp. And the positions of the button needed to close and break the circuit in the prescribed fashion must exhibit the kinematically required continuity. Therefore, $\Delta x \rightarrow 0$ as $t \rightarrow t_1$ and $\Delta x = 0$ at $t = t_1$; that is, the required spatially continuous motion of the base of the switching button issues in the coincidence of the base with E_1E_2 at time t_1. Hence the circuit is predictably closed at time t_1—that is, the lamp must be *on* at the termination of the unit time interval $t_1 - t_0$.

Indeed there is an important respect in which the motion of the button can be understood on the basis of the model of the legato runner's traversal of a progression of Z-intervals in the Dichotomy. The button's $\aleph_0$ downward motions involve the traversal of a total space interval of length

$$L_d = \sum_{n=0}^{n=\infty} \frac{1}{2^{2n+1}} = \frac{4}{6}.$$

And the button's $\aleph_0$ upward motions involve a total spatial displacement of length

$$L_u = \sum_{n=1}^{n=\infty} \frac{1}{2^{2n+1}} = \frac{1}{6}.$$

But, of course, the button actually moves alternately down and up, starting at $t = t_0$ with a downward motion through the initial distance ½. And, after traversing this initial distance, it traverses *twice* each of the space intervals

$$\frac{1}{2^{2n+1}} \quad (n = 1, 2, 3, \ldots)$$

by executing first an upward and then a downward motion through the same interval. Hence the button has the task of traversing a total interval of length

$$L_d + L_u = \frac{5}{6}$$

by traversing first an interval of ½ and then an infinite progression of subintervals

$$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots, \frac{1}{4^n}, \ldots \quad (n = 1, 2, 3, \ldots).$$

Our mention of the legato runner's traversal of a progression of subintervals in the Dichotomy does not overlook the fact that all of the legato runner's $\aleph_0$ Z-runs to his destination point are spatially in the same direction and proceed without velocity fluctuations, whereas the suitably decreasing motions of the button which terminate in a closed state of the circuit at time t_1 are $\aleph_0$ alternately down and up motions. Specifically, the legato runner's uniform motion involves only one initial acceleration of particular finite magnitude and one final deceleration of specific finite magnitude, whereas the accelerations (though *not* the velocities!) of the button increase indefinitely. And, as Wesley Salmon has noted illuminatingly, except for the legato motion, all the processes we are discussing involve indefinitely large *accelerations*. But despite this difference between the legato motion and all the rest, there is a crucial similarity between them, with respect to which the completability of the latter is no less intelligible than that of the former.

For what matters is that the runner reaches his destination at time t_1 after traversing a *progression* of Z-runs, while there does not exist any last Z-run in the progression by means of which the termination of the motion could be effected. And what matters especially is that the terminal instant t_1 of the motion does not belong to any member of the progression of temporal subintervals corresponding to the $\aleph_0$ Z-runs, although every other instant of the motion belongs to at least one such member. In short, what matters is that the runner's arrival at his destination at time t_1 does not belong to any of the Z-runs and is surely not effected by the transversal of a nonexistent *last* Z-run terminating in that arrival. Similarly, the down-and-up motions of the lamp button form a suitably decreasing progression which issues in an on-state at time t_1, even though that on-state is *not* the terminus of any *continuously downward* motion of positive duration during which the lamp would be off. Nor can the on-state at time t_1 belong to any continuous on-state of positive duration whose first instant terminates a single continuously downward motion. Thus, if t_1 were the start of a continuous on-state of positive duration, the particular instantaneous on-state at t_1 would not be the terminus of a continuously downward motion. By contrast, within the confines of the half-open time interval $t_0 \leqslant t < t_1$ *before* t_1, the first instant of any continuous on-state of positive duration *is* the terminus of a continuously downward motion of positive duration. And, again within the confines of that half-open interval *before* t_1, any instantaneous on-state which separates two continuous off-states is likewise the terminus of a continuously downward motion of positive duration.

These considerations enable us to see that the production of Thomson's $\aleph_0$ on-off states would not be feasible under the following alternative switching arrangement, even if we were to ensure the finitude of the total spatial displacement of the switching button (*10*). Let our modified switch button be movable through a linear space interval which is divided by a middle point C into upper and lower segments. And let us assume that the coupling between the positions of the base of the button and the lamp circuit is such as to satisfy the following conditions: (i) when the button base is at any point in the upper segment, the circuit is open and the lamp is off; (ii) when the button base is at any point in the lower segment, the circuit is closed and the lamp is on; (iii) when the button-base-point coincides with the midpoint C, the lamp is on or off, respectively, if it arrived at C from above or from below; and (iv) if the button base is at the midpoint C at a time t, the existence of an on-state of the circuit at t *requires* that the button base reached C from above at or before t, and—unless a circuit component (for example, the lamp filament) has burnt out—the existence of an off-state at t while the base is at C requires that the base reached C from below at or before t. To assure the kinematically required finitude of the total spatial displacement of the button base during the allowed finite time $t_1 - t_0$ (1 minute), let the button base journey back and forth across C $\aleph_0$ times, so as to traverse suitably decreasing distances and reach C at time t_1. After t_1, we leave the switch in the position which it attained at t_1.

Our previous considerations now enable us to assert that, at time t_1, the button base *cannot* have reached C either by a continuous approach from above or by a continuous approach from below. For this much is required if the execution of Thomson's jabbing instructions is to be kinematically feasible. But, in that case, the posited conditions governing the coupling of the switching button to the lamp circuit entail the following conclusion: if the lamp circuit is still intact at t_1 and thereafter, then the lamp is neither on nor off at time t_1 and thereafter! Yet if the lamp circuit is intact at that time, the lamp must be either on or off. One can easily observe which of these two states prevails at t_1 and thereafter by looking at the lamp bulb. Even if the lamp filament should have burned out at time t_1, we can replace the bulb by a new one at that time and observe the state of the bulb thereafter. Hence if it is claimed that Thomson's required $\aleph_0$ on-off states of the lamp circuit permit the lamp circuit to endure intact up to and beyond the instant t_1, a contradiction is introduced by the assumption that these $\aleph_0$ states can be produced by the suggested *modified* switching arrangement.

Thus, in the case of the modified switching arrangement, no less than in the case of the kinematically impossible upward and downward motions discussed initially, the inability to predict the state of the lamp circuit at time t_1 is not at all a matter of insufficient information. And we see that the impossible modified switching arrangement (S_2) differs from the arrangement (S_1) yielding a predictably closed circuit at time t_1, as follows: in the case of S_1, on and off respectively involve coincidence and noncoincidence of the button base with E_1E_2, while the on-state at t_1 is *not* the terminus of any continuously unidirectional (for example, downward) motion; but arrangement S_2 requires that the one lamp state associated with the base's center position at time t_1 and thereafter be the outcome of a continuously unidirectional motion terminating at time t_1. And precisely this is ruled out by the kinematics required for Thomson's process.

Physical Reasonableness of a Dense Temporal Order

The mathematical theory of motion can be vindicated in the face of the James-Whitehead charge of temporal unintelligibility. And the key principles for the required account of the denseness of the time of physics are the following: (i) the spatial path of a moving classical particle is a linear continuum of spatial points, and (ii) the classical particle cannot be at two different spatial points simultaneously. But these two principles entail that, for each of the densely ordered spatial points, there be at least one distinct instant of time at which the particle is at that spatial point. And in the simple case of a particle that visits any given place only once, the temporal betweenness of instants will simply correspond to the dense spatial betweenness of points. I must omit here the numerous technical details of the definition of temporal betweenness which I have given elsewhere (*11*) by means of the two stated principles. But I hope that enough has been said to indicate that this definition of temporal betweenness for punctal physical events yields a dense temporal order, for the following reason: it entails the conclusion that for *any* two events belonging to the motion, there is a linear continuum of others temporally between them.

Indeed, the two principles ingredient in my definition of temporal betweenness serve as a basis for claiming that physical time must be dense because of the way in which space and time are related in physical motion: the assumed continuity of space drives us to the postulation of the continuity (and hence denseness) of time *not* because we are illegitimately "spatializing" time, as Bergson and James thought, but by virtue of our recognition of the role played by spatial continuity in the temporal process of motion. Whatever the *historical* origins of the concepts of denseness and linear mathematical continuity in human thought, denseness is an abstract type of order. And one must therefore not overlook, as Bergson did, the fact that *logically* the attribution of denseness to time no more "spatializes" time than its ascription to space "temporalizes" space. Even if the attribution of denseness to time were shown to be false, this ascription could not then be indicted as a spatialization of time. As well say that the false ascription of denseness to the integers (with respect to magnitude) would constitute a spurious spatialization of these numbers.

It is now clear that, just as a theoretically canonical appeal to such sensed attributes of physical objects as hot and cold can be scientifically stultifying, as Galileo recognized, so also the discreteness of perceived happening can be unjustifiably invoked to encumber theoretical science. And now that we have overcome the objection that temporal denseness is physically unintelligible by justifying its postulation, we see (i) that we are absolved from the necessity of answering "how" a succession of events can occur by exhibiting a discrete sequence of occurrence, and (ii) that there is no ordinal question as to "how" an interval of physical time can elapse at a given spatial point or in the form of a motion *despite* its denseness. On the contrary, having freed ourselves from the intellectual shackles of canonical adherence to perceived temporal happenings, we see that the kinematic answers to such how-questions involve an appeal to the denseness of time. And we see, furthermore, that there is nothing paradoxical about it.

Summary

The mathematical physicist Hermann Weyl (*2*) has claimed that, unless machines can accomplish an *infinite* sequence of distinct operations in a *finite* time, the standard mathematical theory of motion is beset by one of Zeno's kinematical paradoxes. Hence I have compared the kinematics of several such "infinity machines" to the kinematics of the continuous motion of Achilles. And I have argued that, while some designs for infinity machines are indeed inconsistent, others are not impossible on purely kinematical grounds. This argument was coupled with several reasons for denying Zeno's and A. N. Whitehead's allegation of paradox against the mathematical description of the motion of Achilles.

References and Notes

1. J. F. Thomson, *Analysis* **15**, 1 (1954). A more recent version of this article appears in *Zeno's Paradoxes*, W. C. Salmon, Ed. (Bobbs-Merrill, Indianapolis, in press). For an acute critical discussion of Thomson's original article, see P. Benacerraf, *J. Phil.* **59**, 765 (1962).
2. H. Weyl, *Philosophy of Mathematics and Natural Science* (Princeton Univ. Press, Princeton, N.J., 1949).
3. P. W. Bridgman, *Rev. Intern. Phil.* **3**, 490 (1949).
4. A. Grünbaum, *Modern Science and Zeno's Paradoxes* (Wesleyan Univ. Press, Middletown, Conn., 1967), chap. 3; readers outside the United States and Canada are referred to the separate 1968 edition published in London by Allen & Unwin.
5. See G. Vlastos, *J. Hist. Phil.* **4**, No. 2, 95 (1966).
6. G. J. Whitrow, *The Natural Philosophy of Time* (Nelson, London, 1961).
7. In thus using everyday words such as *doing* and *things* in technical contexts, we are fully alerted against such confusions as misidentifying a run_b as a run_a no less than when we use technical terms such as *work* and *energy* in physics. We need to use language to describe the physical process constituting the staccato runner's traversal of the total interval. And in determining whether this process can occur in a finite time, as described, we need to heed the commitments of ordinary language only to the extent of guarding against being victimized or stultified by them.
8. For a discussion of such functions, see E. W. Hobson, *The Theory of Functions of a Real Variable* (Dover, New York, 1957), vol. 1, pp. 280, 300–301, 325.
9. C. H. Chihara, *Phil. Rev.* **74**, 86 (1965).
10. I am indebted to Allen Janis for having concocted this alternative switching arrangement and for having pointed out instructively that it cannot produce Thomson's process.
11. For a full statement of these details, see A. Grünbaum (*4*), chap. 2.
12. I am indebted to Allen Janis, Albert Wilansky, and Wesley Salmon for very helpful discussions of some of the "infinity machines," and to the National Science Foundation for support of research.