# Letters

## College Math for 11-Year-Olds

A study to determine whether fundamental mathematical concepts considered suitable only for college students can be absorbed by 11-year-old children was conducted with a group of ten sixth-graders selected at random from the John D. Pierce Laboratory School at Northern Michigan University. These children met with me for 3 half hours a week for 8 weeks and we touched upon set theory, functions, matrices, and linear transformations.

We began by describing a *set* or *collection* using the terms and symbols for: *meet, join, complement, implies, element of, and, or, not*, and *difference*. With colored chalk and crayons we made rough maps of the Upper Peninsula of Michigan, Wisconsin, counties in those states touching Lake Superior, cities as points in the counties, and counties not touching Lake Superior. We described all possible combinations and cross-combinations in set jargon, placing emphasis on the definitions. Before long the students proved a few simple theorems such as: (i) complement of meet = join of complements; (ii) $B$ is contained in $C$ and $b$ is a point in $B$ implies $b$ is a point in $C$; and (iii) $B$ is contained in $C$ implies $B$ meet $C = B$. Enthusiasm was uniformly high. One girl almost immediately proved the first theorem in approximately the following manner: "*Meet* means you're in both sets. *Not meet* means you missed at least one set. So you are in the complement of some set—any set. So you're in the join of all the *nots*." This is completely rigorous.

By the third week we had defined *domain* and *range* sets and the rule which connects each point in the domain set to one or more points in the range set. We illustrated a *single* and a *multiple valued function* and *inverse functions* when they existed by the "phone book" figure of speech. People's names constitute the domain and ten-digit numbers (a couple of smart kids thought we ought to include area codes) form the range. A person could have two phones in his house or two people could share a phone and we could have a city where everybody had only one phone and no two people shared a phone—then it would be easy to look up who owned each ten-digit number. Thus in the case of a 1 to 1 function, we could have what is called an inverse function *uniquely determined*, and that expression is now in the children's vocabulary. We applied this to linear functions and these sixth-graders soon could sketch such lines as $y = 1 + 2x$. One girl decided she could, however, draw $y = x^2$. She thus introduced us to parabolas.

Then we drew a few hyperbolas of type $xy = \text{constant}$, the constant being usually an integer with easily spotted prime factors, such as 70 or 24. Another girl was the first to realize that the rectangles whose diagonals run from any point $(x,y)$ on the hyperbola to origin are all of the same area.

Our group next began to study matrices. The definition of a matrix, in which the elements need not be numbers, was easy. One girl realized that the way the children were sitting in their class in a rectangular formation was a matrix—they, the children, being the elements. If the elements happen to be numbers, then the concepts of addition, zero matrix, and negative follow immediately. A boy demonstrated addition of two matrices showing how this is impossible unless the two matrices are exactly the same size and shape. A girl pointed out the effect of adding the zero matrix; that is, nothing happens. Multiplication of matrices was, of course, difficult for them, but eventually all of the group were able to understand the row-by-column rule, and that it was by no means similar to the element-by-element rule used in addition. We ultimately employed 2 by 2 matrices for the purpose of making linear transformations in Euclidean two space. We took various shapes in the $(x,y)$ plane and, by using the 2 by 2 matrix as a "magic wand," transformed them into other shapes in what we named the $(u,v)$ plane. For example, one boy would make up a 2 by 2 matrix, then we would all take lines that are easy to work with (horizontal, vertical, or through the origin), and see what became of that line by waving the "magic wand." For example: Let matrix

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$$

Then let matrix $z = (x \;\; 2)$ represent horizontal line $y = 2$. Matrix multiplication $zA = (-x + 2, \; 2x + 6)$ shows what happens to this line when acted upon by the magic wand. The children were not algebraically sophisticated to the point of eliminating $x$ in the pair of equations $u = -x + 2$, $v = 2x + 6$ which to the initiated indicates that $v = 10 - 2u$, but they could take any two values for $x$ along the line $y = 2$, plug these values into the $u(x)$ and $v(x)$ formulas, draw a line and then take a random third value of $x$ on the line $y = 2$ as a check to see that the new or third point $(u,v)$ was on that line.

A bright lad discovered how a particular 2 by 2 matrix transforms circles and parabolas by plotting about four or five points on each, performing the multiplication (or, as we said, waving the magic wand), and plotting the new points on a second or $(u,v)$ graph; it was apparent after a while that parabolas and hyperbolas could sometimes fall into circles or ellipses or vice versa. "You never can tell about these 2 by 2 matrices. They do funny things."

Finally, the students were surprised when a girl made up a matrix that proved to be singular. Every shape in the $(x,y)$ plane came out on one straight line. We all tried and tried, but no matter what curve we started with in the $(x,y)$ plane, all came out along that straight line. On this note we ended the study.

In conclusion, sixth-grade students can understand fundamental concepts of analytical college-level mathematics, apply them to some extent to the world of science as they see it, and enjoy it more than conventional elementary arithmetic. It is worth continuing such an experiment at fourth or fifth-grade levels. The possibility of incorporating this type of "new math" into the regular elementary curriculum is worth consideration.

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