One *can*, however, conceive of a universe in which the test body is totally force-free while its velocity is measured as a function of time. Let all the matter in this universe be disposed in concentric, uniform, spherical shells surrounding the test body. Then the test body is force-free as long as it stays within the otherwise empty space inside the smallest shell.

The only remaining conceptual problem consists of making the measuring instruments into spherical shells. (For example, optical lenses cannot be employed without disturbing the spherical symmetry.) The following arrangement seems to meet the requirement. The inner shell is thin and black, except for pinholes at two or more widely separated locations. A transparent substance of equal density fills the pinholes so the gravitational uniformity of the spherical shell is not disturbed. The next shell is transparent and thick (it could even be empty space) and is surrounded by a layer of photographic emulsion. The test body contains a clock which controls the periodic flashing of an isotropic light. (Uniform radiation in all directions is necessary so that the test body experiences zero net-radiation force.) Thus, if the motion of the test body relative to the shells is in a straight line at constant speed, equally spaced rectilinear (after due correction for the geometrical optics involved) images will appear in the photographic emulsion behind each pinhole. After the experiment is completed, the observer comes in from infinity and determines the positions of the images.

Under the conditions of this "thought experiment" the test body is rigorously force-free while its velocity is recorded, thereby refuting Hanson's statement that "the counterfactual character of Galileo's law stands not merely as an observation that no bodies *are* found to be force-free but, rather, as a consequence of there being no body whose motion is uniform and rectilinear which could possibly be force-free. Any alternative crushes the gravitational cornerstone of mechanics. Appraisals of the law's logical status are pierced by this point. The law thus refers to entities *not* such that, although never observed, they remain observable but, rather, entities that are unobservable in physical principle."

MARVIN M. MUELLER
*307 Manhattan Loop,*
*Los Alamos, New Mexico*

The ingenious thought experiment suggested by Mueller is doubly admirable: it is provocative and instructive intrinsically, and it indicates the need for greater clarity in the exposition of my work on Galileo. It was not my intention to deny the possibility of conceiving of a force-free body in general, but rather only in the kind of universe we actually inhabit. The concept makes sense; how else could I meaningfully deny that it has any application in this world? My point was that we could imagine nothing which would give the concept any application whatever in *this* world, in Galileo's.

Mueller shrewdly delineates another universe within which all the constituent matter is arranged in spherical shells, nullifying thereby any unbalanced gravitational force acting on a test body within the innermost shell. Of this thoughtful theoretical thrust, however, three things can be said, I think.

1) Mueller has changed the conditions of the thought experiment fashioned in my paper. This I invited him to do by failing to indicate that my construction was bounded by the known properties of this universe, given which one could not even imagine a body as force-free.

2) Archimedes cried out for a fixed platform from which he could lever away the world. He did not deny that from a second platform his own original "thought platform" could itself be levered away. Mueller has pointed out that the assumptions we make in studying the dynamics of this world could themselves be demonstrated in some other world; but this does not establish such another world to be assumption-free. Consider Mueller's own presuppositions: (i) that light corpuscles (such as those emitted from his test body) traverse Euclidean straight lines, and (ii) that these corpuscles traverse equal areas in equal times. Without these assumptions Mueller, coming "in from infinity," could not infer from equally spaced rectilinear images in the emulsion an inertial motion in the test body. (Notice the phrase "after due correction for the geometrical optics involved"; his assumption of rectilinear, uniform trajectories for all light corpuscles is obvious here.)

3) In Galileo's world Mueller's assumptions are unnecessary. The kinematical properties of photons can be proved, not just presupposed, by demonstrating that light corpuscles traverse paths *that would be traversed by force-free bodies*, if they existed. Mueller's think-tank employs a different springboard; he demonstrates that force-free material particles (in his spherical universe) traverse paths that *would be traversed by light corpuscles.* He assumes what Galileo could show; Galileo assumed what Mueller can show. But whoever undertakes to show anything whatever must assume something or other—that is part of the logic of proof. Galileo and Mueller both feel it makes sense to think of "force-free bodies." But, given Galileo's other commitments, it does not represent a genuine possibility in this universe, as I tried to show. Mueller transforms the idea into a genuine possibility by changing his universe.

N. R. HANSON
*Department of Philosophy, Yale University, New Haven, Connecticut*

## Numerical Analysis: Pure or Applied Mathematics?

The exchange of views (Letters, 16 July, p. 243) subsequent to Hamming's article "Numerical analysis vs. mathematics" (23 April, p. 473) raises a number of interesting points on the relation between science and mathematics and whether numerical analysis should be viewed as being closer in spirit, outlook, or working methods to one or the other. In my view, all parties to the dispute have come close to, but seem to have missed, what may well be some essential aspects of the relations between the three fields (or two, if one wishes to assign numerical analysis to one of the others).

An important distinction between science and mathematics, which is appreciated by all concerned, is the empirical content of the former, that is, the appeal to experiment, and the purely formal structure of the latter. One might make a comparison with language; the rules of grammar or syntax permit us to distinguish between grammatical and ungrammatical sentences, but they can make no statements about the world of phenomena and everyday objects which we use the language to discuss. Experience, observation, or experiment can be viewed as providing the semantic content, or making meaningful the vocabulary that is used to discuss the world of science, while mathematics, as the "science of structure," provides the syntactical rules permitting the formation of "grammatical" scientific sentences (and the rejection of ungrammatical ones). From this point of view it is a matter of historical accident whether a particular mathematical concept arises in response to needs of scientists or develops from within pure mathematics and is later applied in science. The "crucial sentence" of Hamming in which the lack of need for *empirical* verification of mathematical concepts leads to the conclusion that much of modern mathematics is more closely related to medieval scholastic arguments than to science seems quite irrelevant to the main point at issue. *All* mathematics has that characteristic; science makes extensive use of those branches of mathematics whose postulates are in fact verified (often in such an extremely indirect manner, however, that using a strong word like "verify" may not be appropriate). In the growth of theoretical science one is continually presented with new situations in which new kinds of postulational systems must be employed.

The mathematicians, by providing the scientists with the fruit of their labors, permit a wide choice of new formal tools with which new attempts to organize experience can be made. The symbiotic relationship between science and mathematics, in which science uncovers problems that can inspire new developments in mathematics, and mathematics develops formal systems which accelerate the progress of science, is so well appreciated and so fantastically fruitful that it is hard to imagine anyone trying to "legislate" away any of the essential freedoms so helpful in the past.

The real issue in the dispute, I believe, is the nature of numerical analysis. Is it pure mathematics, or is it a field of applied mathematics close in spirit to the sciences? Numerical analysis generates desired arithmetical or other mathematical data in a well-defined mathematical system (for example, a set of partial differential equations with given boundary conditions) by mathematical methods. It is thus "all mathematics," in contrast to theoretical physics, where the nub of the problem is the discovery of a formal system which adequately describes experience. Introduction of constraints (such as minimum cost or errors) does not change the fact that the problem is fundamentally mathematical.

Whenever a sphere of applications develops which makes demands on a particular mathematical discipline which go beyond the state of development of the field at the time, the "customers" frequently proceed to remedy the deficiency from the point of view of their application rather than from the point of view of pure mathematics. If these "customers" are scientists or engineers, for whom the empirical has much importance, they will tend to neglect rigor, elegance, generality, and even consistency, in their attempts to get their main jobs done. From Hamming's article it would appear that this has occurred in numerical analysis, and that the "customers" may have tended to dominate the field in recent years because the pure mathematicians were otherwise occupied. It may be that because of history and tradition this will continue for some time (or even permanently). If I may venture to prophesy, however, I predict that the challenge of developing the mathematics of numerical analysis on a rigorous basis in keeping with the standards of pure mathematics will eventually be taken up. The subtle logical and combinatorial problems associated with computers, switching networks, systems design, and numerical analysis have so many interesting and important facets both from the viewpoint of applications and from that of pure mathematics that I cannot see the pure mathematician forever ignoring these fields. I also do not see how the "customers" will make much headway on many important problems unless some of them become, in effect, highly competent pure mathematicians. This kind of thing has often occurred in the past, and will probably happen many times in the future. Ultimately the development of numerical analysis as a "science" can be expected to encompass construction of a solid basis, in the sense of pure mathematics, similar to what has occurred in statistics. Modern probability and statistics, with their rigorous measure-theoretical basis, are a far cry from the simple data collection and reduction of the past. Along with the development of the "pure" basis has come a tremendous increase in the power and scope of statistical methodology and of its usefulness in applications. We expect to see a similar evolution in numerical analysis.

JEROME ROTHSTEIN
*Laboratory Electronics, Inc.,*
*1075 Commonwealth Avenue,*
*Boston, Massachusetts*