Numerical Analysis vs. Mathematics

Modern mathematics often does not deal with the practical problems which face numerical analysis.

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The purpose of this paper is to illustrate by means of examples some differences between numerical analysis and mathematics. These differences may be loosely classified under the headings of taste, importance of processes, misleading models, and the effects of "noise," though some of the examples fall under several headings.

One reason for pointing out these differences is that just as statistics some 30 to 40 years ago was regarded as a dirty, inferior part of mathematics, so too at present numerical analysis is often thought to be an inferior part of mathematics, a part which should strive to improve itself by copying mathematics. And, just as statistics broke free and began to develop along its own natural lines, so too it is hoped that numerical analysis will soon follow its own natural growth.

The reason for using examples is simply that neither mathematics nor numerical analysis is defined in any satisfactory manner, hence no direct proof of the differences can be given. I hope that the examples I use will be regarded as typical of many others and not as special isolated cases.

Experience shows that many persons regard these examples as an attack on and a criticism of mathematics rather than what they are intended to be, merely illustrations of some differences between mathematics and numerical analysis. It should also be pointed out that just as there are still many mathematicians writing about mathematical statistics, so also many mathematicians write about mathematical numerical analysis—it is not so much the subject matter that makes a field but rather it is the attitude toward the material that serves to define the field.

Mathematical Taste

Let us begin with "taste." Mathematicians (quite properly) attach much importance to elegance, deep results, important theorems, and so forth. However, mathematicians tend to identify elegance with surprise, and hence to arrange their final presentation in a surprising manner rather than in a manner which would tend to reveal how it was found. As a result the poor student merely enjoys the elegant presentation without finding out how to do mathematics. I am hopeful that numerical analysis will not go down the path of elegance as opposed to clarity.

As an illustration of this point, consider the widely cited result that $\sqrt{2}$ cannot be written as a fraction. It sounds somewhat deep and perhaps difficult to prove. But if I say, "The square of a fraction (in reduced form) cannot be an integer," you see almost immediately not only the isolated result for $\sqrt{2}$ but the general case, and furthermore you have no difficulty in making the trivial generalization to other integer powers. It is only by stating the theorem backwards, as it were, that the surprise is obtained.

As a second example of taste, mathematicians tend to put great emphasis on existence theorems whose purpose is to show that what they are talking about actually exists. Unfortunately, all too often the method of proof is non-constructive, so that the person who wishes to do something has no idea of how to find the solution that has been proved to exist. On the other hand in practical computing we often compute things when we have no existence theorem to show they exist.

As a third example of taste, suppose I announce that some triangles, though which ones I have no idea, have a certain property. Most mathematicians would quite justly look at me askance. But if in numerical analysis I give a method which often, though I do not always know when, produces a solution to a difficult problem, then it is likely to be regarded as an important advance.

Attention to Processes

Let us turn to the second point, the importance of processes. On the surface of mathematics one sees great care in the statement of theorems and in the rigor of the proofs, but it is surprising how sloppy has been the treatment of processes. You are surprised? Consider, then, the command given to generations of algebra students, "simplify the following:" What does "simplify" mean? Quite recently a definition for 9th grade students has been hammered out: the simplified result should have only one division and no radicals in the denominator. In short,

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

is not simple, but

$$\frac{\sqrt{ab}\ (\sqrt{a} + \sqrt{b})}{ab}$$

is the corresponding simplified form!

It must be obvious without my producing more examples of this kind that there are probably many, as yet apparently unanalyzed, different meanings to the oft used word "simplify," and also that the different meanings probably have different domains of applicability.

Let me pick another well-known result, this time from the theory of equations. One way to solve a real quartic requires finding the roots of a resolvent cubic. The real quartic can always be factored into the real quadratics, but in the case of four complex

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roots only one of the roots of the cubic will do this. Which one? Mathematicians apparently seldom care, nor perhaps should they, to answer this question, but to the numerical analyst it is of great importance if he is to use this method on a machine.

More generally, when one attempts to put many of the well-known processes of mathematics on a computing machine one finds that there is a great vagueness, and waving of hands, and occasional shouting of "Any fool knows!" and that in the long run a much more careful examination of the basic ideas and processes must be made before one can make much progress. I have been repeatedly shocked to find out how often I thought I knew what I was talking about; but that in the acid test of describing explicitly to a machine what was going on I was revealed to have been both ignorant and extremely superficial. It is this many-times-repeated experience that has led me to assert that mathematics has often chosen to ignore the careful examination and exposition of the methods it uses.

Let me repeat, I am not saying that the mathematicians were wrong or superficial; I am documenting the point that the two fields have different goals and objects—mathematics has tended to be precise in its statements of results and in its rigor, while numerical analysis, and computing generally, tends to put great emphasis on the clear statement of the processes used. This emphasis on methods used is receiving more attention in many fields of activity, and thus, in some respects, the goals and objectives of computing are more in tune with the rest of our scientific culture than are those of traditional mathematics.

Mathematics and the Real World

In using the phrase "misleading models" it is difficult to avoid the accusation that it is I who am making the mistake rather than the mathematician who published the model. Yet I feel that I have the reasonable point that many of the most famous results in mathematics are regularly announced as if they were relevant to the real world.

Let me give as a first example a widely advertised result in pure mathematics—the impossibility of trisecting an arbitrary angle with straightedge and compass. The proof rests on the postulate that a line is determined by two points. If this postulate is modified slightly so that a straightedge with two marks is allowed to define a straight line, then there are well-known constructions which do trisect an arbitrary angle. The amount of adjustment necessary in the case of the marked ruler is not significantly greater than that for the two points. Of course, trisection by repeated guessing is also very easy in practice. The theorem is simply an artifact of a small detail of a postulate.

Let me cite another example of the irrelevance of some mathematical results of practical applications. The Riemann integral exists for functions which are continuous except for, say, having 10^{78} discontinuities. Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, that whether, say, an airplane would or would not fly could depend on this difference? If such were claimed I should not care to fly in that plane!

Since I am apt to be misunderstood at this point let me clearly state: I am not commenting on mathematics, I am commenting on the relevance of much of mathematics to computing, and more broadly to the application of mathematics to the real world. I am attempting to document my thesis that the objectives, standards, and so on of mathematics are often not appropriate to computing and to many of the applications of mathematics. I am also trying to show that the computing expert needs to be wary of believing much that he learns in his mathematics courses; in a sense he must learn mathematics so well that he can defend himself against it.

As another example of what I mean by "misleading models," there is in a readily available book a lovely treatment of the relative effectiveness of the rule of false position (or secant method) and of Newton's method (of fitting a tangent line). The treatment is based on the assumption that when a function at a point x_1 has been evaluated, as much computing labor is required to evaluate a derivative at x_1 as to evaluate the function at a new point x_2. As anyone can see who looks at the kinds of functions generally occurring in numerical analysis, once the pieces of the function have been computed it is usually relatively easy to compute the derivative at the same point—the same radicals occur, and the same logs, exponentials, and trigonometric functions tend to occur in both. But when the author finally assembles the comparison of the two methods in one place there is absolutely no mention of the highly unrealistic hypothesis! As a result quite a few people, impressed by his mathematical rigor and elegance, have been led astray in their choice of a practical method.

Generally speaking, in the early history of mathematics long experience in the real world preceded both the abstraction of the postulates and the formulation of the definitions of geometry, and subsequent experience has validated their general usefulness. Thus early mathematics tended to follow the classical test of science, the regular (though not exclusive) appeal to observations in the real world. But it is difficult to imagine how by appeal to observations many of the postulates of current mathematics could either be verified or shown to be unsuitable, and one can only conclude that much of modern mathematics is not related to science but rather appears to be more closely related to the famous scholastic arguing of the Middle Ages. It is my belief that numerical analysis will be wise to follow the lead of the sciences, and that its usefulness, and hence its ultimate health, will probably be best served by regular, though certainly not exclusive, appeal to the world of experience.

"Noise"

The last point to be considered is the effects of "noise," that is, small uncertainties in the initial data, in the model, or in the processes used. In a sense the trisection of an angle with ruler and compass is an example of large effects due to small changes in the assumptions. It should be obvious that in practical situations only "noise-resistant" mathematics (statisticians say "robust") can be used, and the rest of mathematics should be classed as "art forms" of pure mathematics.

Let me start with the fundamental theorem of algebra as an example. In some respects the main point of the theorem is that the functions 1, x, x^2, . . . , x^n are linearly independent in any interval—otherwise a linear combination would be identically zero in the interval. To fix ideas let us take the interval ($-1 \leq x \leq 1$). Now, almost all computing done on a computing machine is to a fixed number of places, and as a result there is roundoff and what is sometimes called "a roundoff noise level." In principle the roundoff errors are perfectly determinate, but it is convenient to regard them as random. Suppose this "noise level" is of size 10^{-6}. Then, as Chebyshev proved long ago, there exists a polynomial which starts out with x^{21} (having unit coefficient) and which is less in absolute value than 10^{-6} in the whole interval. What then do we mean when we assert that $1, x, x^2, \ldots, x^{21}$ are linearly independent in the interval ($-1 \leq x \leq 1$)? We cannot distinguish the values of the polynomial from zero. A polynomial of degree 21 having the leading coefficient equal to 1 can be changed to a polynomial of degree 20 and we cannot hope to detect the difference!

It is a simple matter to convert this special case to any interval and any noise level—given them, it is easy to determine the lowest-degree polynomial with the leading coefficient 1 such that the polynomial is always less than the given noise level in the given interval. Thus the Chebyshev polynomials measure, in a sense, the breakdown in practice of the fundamental theorem of algebra. Thus, we see that in numerical analysis we cannot glibly invoke this theorem without digging much deeper and asking if it is relevant and appropriate to the given situation. The mathematical theorem asserts that x^{1000} and x^{1002} are linearly independent in any interval no matter how small, but in practice how impossible it may be to detect!

Again, a mathematician may prove that some iterative scheme converges, but unfortunately on a computing machine we may come down to a loop where *a* produces *b* and *b* produces *a* on successive iterations. How big shall we make this "circle of confusion?" Much depends on the development of roundoff in the loop, and much on the mathematical structure of the loop itself, and these are not easily analyzed in advance.

Let me generalize from these isolated examples and assert that much of mathematics has been concerned with ideal "noise-free" concepts, and that in the practical applications of mathematics only those theorems and results which are "noise-resistant" can be of much use.

Numerical analysis is dominated by the simple fact that our machines have a finite number length, be it single, double, or triple precision, and that almost all the numbers and processes we use will involve "roundoff noise." Lest you regard this as a defect let me point out that in quantum mechanics there is the famous Bohr correspondence principle which in a sloppy form states that, as the quantum size goes to zero, quantum mechanics must pass over to classical mechanics. Similarly, there is the obvious principle that as the roundoff noise level approaches zero the results of numerical analysis must pass over to classical analysis. And just as quantum mechanics is far richer in effects than is classical mechanics, so too is numerical analysis far richer than classical analysis. It only requires the courage to exploit and develop these new effects.

It is not easy to propose a simple program for modifying mathematics to fit the needs of people in numerical analysis. So many of the basic ideas are not appropriate. For example it is customary in mathematics to say that if $A = B, B = C, C = D \ldots Y = Z$ then $A = Z$; the idea that equivalence is indefinitely transitive simply breaks down in many practical applications of mathematics.

Again, the real number system of mathematics has many properties which are not mirrored in reality. For example the "real numbers in a computer" are finite in number, have about as many numbers between 0 and 1 as there are above 1 (in a floating point machine), and are not, obviously, equally spaced.

Summary

I hope I have shown not that mathematicians are incompetent or wrong, but why I believe that their interests, tastes, and objectives are frequently different from those of practicing numerical analysts, and why activity in numerical analysis should be evaluated by its own standards and not by those of pure mathematics. I hope I have also shown you that much of the "art form" of mathematics consists of delicate, "noise-free" results, while many areas of applied mathematics, especially numerical analysis, are dominated by noise. Again, in computing the *process* is fundamental, and rigorous mathematical proofs are often meaningless in computing situations. Finally, in numerical analysis, as in engineering, choosing the right model is more important than choosing the model with the elegant mathematics.

I believe that it is important to make these distinctions, not only for numerical analysis, but also because they are important for the debate on what kinds of mathematics should not be taught; also because the failure to do so has, on occasion, caused government money appropriated for numerical analysis to be diverted to the art form of pure mathematics.