Water Waves and Hydrons

The term "hydron" is proposed for fictitious particles that travel with the group velocity of waves.

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In modern physics the word *wave* has magic in it, but no one can say just what a wave is. Correction: everyone thinks that the only waves worthy of the name are those in which he is interested. Let us abandon such selfish views and cast a glance over those physical phenomena that are commonly called "wave phenomena." Here is a list of the most important.

1) Water waves (surface gravity, capillary).

2) Sound waves in air or other gases (including blast waves).

3) Elastic waves (violin string, bell, earthquake).

4) Electromagnetic waves (radio, television, radar, infrared, visible light, ultraviolet light, x-rays, gamma rays).

5) Matter waves (of de Broglie, Schroedinger, Dirac).

6) Gravitational waves.

What is common to them all? Students are told that waves transmit energy without the transmission of matter, and that is true of all the waves listed above, with a signal exception—matter waves. But there is a certain mystery about this transmission of energy, because it is widely recognized that the energy which is transmitted travels, not with the speed of the waves, but with the group velocity. Furthermore, when energy is emitted or absorbed in microscopic quantities, this emission and absorption take place as if the energy traveled, not in waves, but in particles of energy. This concept of parcels of energy is clearest in the case of electromagnetic waves, and the word *photon* is now an accepted word and concept in physics; it was invented in 1926 by G. N. Lewis, who wrote: "I therefore take the liberty of proposing for this hypothetical new atom, which is not light but plays an essential part in every process of radiation, the name *photon*."

For fictitious particles carrying energy in elastic vibrations the word *phonon* is used, and the word *graviton* is ready for the description of the transmission of gravitational energy, although the corresponding physical phenomenon has yet to be observed.

It may be noticed that in the list of wave phenomena the word *ray* occurs twice, in x-rays and gamma rays. If we regard this as a mere historical survival and speak instead of x-waves and gamma waves, as indeed we might, then the word *ray* disappears entirely from view; the ray is swallowed up in the wave. It would be wrong, however, to let the word *ray* disappear from physics, for the great antithesis of the century is better set out in the words *wave* and *ray* than in the more usual *wave* and *particle*.

Nature Seen through Half-Shut Eyes

A man who takes a magnifying glass into a picture gallery and examines the canvases at a distance of 3 inches may acquire much interesting information about the texture of paint, but he does not see the pictures. It is better to stand away. If trivial details still intrude, it is better to half-shut the eyes. As a final step, it is well to shut the eyes completely and think about what has been seen.

This has a moral for science, particularly in an age when technical skill in observation is rapidly advancing. Had the accuracy of observation been greater in his day, Kepler would have found it harder to reach the conclusion that the planets pursue elliptical orbits round the sun. He did not have to half-shut his eyes; they were half-shut for him. But once a concept has been understood in simple terms, details provided by improved observation can be added without obliterating the central scheme.

Each of the six types of waves listed possesses an elaborate mathematical theory, and there is little in common between these theories. In some, but not by any means in all, we find the classical wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - k^2\frac{\partial^2\phi}{\partial t^2} = 0$$

where k is some constant. But this equation is not really adequate to deal with dispersive waves. In fact, the more closely we examine these various wave phenomena, the more unlike one another they appear, until the only link between them seems to lie in the use of the word *wave*. To see the connection which justifies this common name, we must abandon the magnifying glass and look at the phenomena with half-shut eyes. To do this, we have to throw off our modern sophistication and look at things as they appeared to three great men of the past: Pierre de Fermat (1601–1665); Christian Huyghens (1629–1695); and William Rowan Hamilton (1805–1865). Historical accuracy about their work is of no importance here, for all we seek from them is the blurred images of nature with which they worked.

From Fermat we learn to think about a moving particle, its path determined by a principle of least time, $\delta\int n\,ds = 0$, where a "refractive index" n is supposed known as a function of position, and possibly of direction also. The history of the moving particle may be called a "ray"; no waves appear. Huyghens, on the other hand, described how *waves* propagate by means of secondary wavelets; rays do not appear. Hamilton wove these two concepts into a single mathematical theory, in which one can start from rays and evolve the waves, or vice versa.

Hamilton was a mathematician rather than a physicist, and one might say that he viewed nature through half-shut eyes. But there is nothing blurred about his mathematics, and, if we can suppress the archaic physics in his work and treat it as a mathematical structure, we find in it the goal we are seeking, the link between the several types of waves which seem so different to the specialists who work with them.

Of course, a price must be paid for myopic vision. In applying Hamiltonian methods we shall make mistakes in the sense that we shall say things about nature which are not true. But, on the

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other hand, we find ourselves able to answer many questions otherwise unanswerable by virtue of their apparent complexity, and, if the answers are not quite correct, it is better to have an answer which is at least qualitatively right than no answer at all.

Hamiltonian theory is like an oracle. Some questions about waves it will not answer at all, for its vocabulary is limited, and questions do not make sense to it unless they refer to frequency and wavelength (or wave numbers). If a question is put in such terms, the rough-and-ready rule for assessing the reliability of the oracle's answer is this: it is likely to be good physically if the wavelengths involved are small compared with other relevant lengths, or if the frequency is high relative to other relevant frequencies.

Optics and Water Waves

Of all the phenomena under consideration, optics has been the most studied throughout the history of physics. The crude facts of optics are known to the man in the street, and he thinks in terms of rays of light (Fermat). To explain interference simply, the construction of Huyghens is invoked. The next step is to Maxwell's equations, and somewhere in between we cross a barrier separating "geometrical optics" from "physical optics." The student who crosses this barrier is supposed not to look back at the toys of his intellectual childhood, but he gets two rude shocks. First, if called upon to design an optical instrument, he finds it quite impossible to solve Maxwell's equations for this purpose, and he must fall back on the elementary methods of geometrical optics. Secondly, in dealing with photons in cases of small illumination he finds himself thinking of them as particles in quite an elementary way. In fact, the Maxwellian facade promises more than it can actually achieve, and in ninety-nine cases out of a hundred the method actually used is the method of geometrical optics.

There are no elementary students of water waves. In fact, there are very few students of water waves at all in comparison with the numbers interested in other types of waves. The reason is not far to seek. Ordinary differential equations are much easier to deal with than partial differential equations. Hence, particles are easier to deal with than continua. Optics started with particles of light, the easy way, and only came gradually to "waves in the ether." But on water it is waves that we see, not particles carrying energy. Thus, inevitably water had to be treated as a continuum. Partial differential equations had to be written down as a starting point, and the sophistication of this approach separated the theory from the simple phenomena, observed by the man on the beach, of waves rolling in from the ocean. It seems that only recently has it been realized that water waves can, with fair accuracy, be treated as light waves are usually treated —by the so-called methods of geometrical optics (*1*).

It is a psychological mistake, I think, to call this the method of "geometrical optics," because one is then in danger of reading into it optical shades of meaning. In ordinary optics dispersion plays a very minor role; in a vacuum the speed of light is rigorously independent of wavelength, and in ordinary transparent media the dependence is slight, so that the correction of chromatic aberration involves only small changes in design. But in water waves (and in de Broglie waves, too) dispersion is a major effect and its neglect would be ruinous. Hence I suggest that, in referring to that method which links all waves together, we should speak of the "Hamiltonian method," even though the applications lie far outside those contemplated by Hamilton and the method itself is somewhat generalized (*2*).

The Hamiltonian Method

I have referred above to the relative difficulty of dealing with partial differential equations in comparison with ordinary differential equations. If we examine carefully any type of wave phenomenon we find partial differential equations. The Hamiltonian method is essentially a process of cheating by which we substitute ordinary differential equations. We excuse this cheat by saying that, under certain circumstances, it is only a very little cheat. (Reflect, if you wear glasses, that a thoroughly honest optician would not have them ready yet—he would still be trying to solve Maxwell's equations!)

What follows applies in principle to all the wave phenomena listed earlier, but for notational reasons I must concentrate on water waves. All physical phenomena take place in four-dimensional space-time, and that is true of water waves. But we shall regard water waves as an up-and-down disturbance of a *plane* surface, and so we can get on with only two dimensions of space. We speak of a three-dimensional space-time, in which we have coordinates x_1, x_2 (rectangular Cartesian coordinates in the undisturbed surface) and x_3 (the time).

We start with the simplest type of waves, waves with parallel straight crests, all of the same height. The vertical disturbance ζ of the surface can then be written

$$\zeta = a \cos \psi \qquad (4.1)$$

where a is a constant amplitude and ψ is the phase angle, which may be written

$$\psi = x_1 y_1 + x_2 y_2 + x_3 y_3 + \varepsilon \qquad (4.2)$$

where the three y's are constants and ε is another constant. We recognize at once that y_1 and y_2 are wave numbers, connected with the wave-length λ by

$$y_1^2 + y_2^2 = (2\pi/\lambda)^2 \qquad (4.3a)$$

and y_3 is the circular frequency, connected with the ordinary frequency ν by

$$y_3 = 2\pi\nu \qquad (4.3b)$$

The wave speed (or phase speed) is

$$W = \lambda\nu = y_3/y$$
$$y^2 = y_1^2 + y_2^2 \qquad (4.4)$$

There is no cheat here. The next step is an appeal to sophisticated hydrodynamics. We find, for example, that there is a formula for wave speed in terms of wavelength for water of given constant depth; equivalently, in the foregoing notation it reads

$$\Omega = y_3^2 - gy \tanh hy = 0 \qquad (4.5)$$

g being the acceleration of gravity and h the constant depth; Ω is merely a symbol for the expression written here. Still no cheat. We note further that, from its definition, ψ has partial derivatives as follows:

$$\partial\psi/\partial x_r = y_r \qquad (4.6)$$

the subscript r, here and later, taking the values 1, 2, 3. We may now, if we like, substitute for the y's in Eq. 4.5 the partial derivatives of ψ, and so obtain for ψ a partial-differential equation. (This is, in fact, the famous Hamilton-Jacobi equation of dynamics, in disguise.)

All this is symbol-juggling in preparation for the cheat, which comes now. We take another look at Eq. 4.5 and recognize h as the *constant* depth of the water. But suppose we are interested, as we well may be, in water of variable depth. Then h is not a constant but a function of x_1 and x_2, and we no longer have any right to accept Eq. 4.5 as a valid equation connecting frequency, wavelength, and *variable* depth, because it was given to us only for constant depth. But this is what we do, muttering something about the depth changing only slowly with change of position. It is better not to mutter anything but to live dangerously, accepting Eq. 4.5 with h variable; accepting, too, all the foregoing formulas with one modification in interpretation—the y's are no longer constants, as they were, but variables (conscience mutters "slowly changing").

When, previously, we thought of substituting in Eq. 4.5 from Eq. 4.6, it was in the nature of a joke; we did not have to find the phase angle, for it was given by Eq. 4.2, in which the y's were regarded as given constants. But this substitution is no longer a joke: it gives the formidable-looking partial differential equation

$$\left(\frac{\partial\psi}{\partial x_3}\right)^2 - g\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2\right]^{1/2} \times \tanh\left\{h(x_1, x_2)\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2\right]^{1/2}\right\} = 0 \qquad (4.7)$$

To find the phase angle over the surface of the water and at all times, this equation must be solved. If it is solved, then we have the waves; they are given by equations

$$\psi = \text{const.} \qquad (4.8)$$

These are curves on the water surface, the crests being $\psi = 0, 2\pi, 4\pi, \ldots,$ since phase angle must change by 2π on passing from crest to crest.

Rays and Hydrons

What has just been described is only a first step in the Hamiltonian method. By cheating a little we have obtained a partial differential equation (Eq. 4.7) for the phase angle. The next step is honest and purely mathematical. It consists in solving Eq. 4.7, or any such equation involving only *first* derivatives, given suitable initial data such as the phase for all times on a given curve. Pursuing the method (*3*), we find that all we need to do is to solve a certain set of *ordinary* differential equations, which are in fact of the Hamiltonian form familiar in dynamics (*4*):

$$\mathrm{d}x_r/\mathrm{d}w = \partial\Omega/\partial y_r, \qquad \mathrm{d}y_r/\mathrm{d}w = -\,\partial\Omega/\partial x_r \qquad (5.1)$$

($r = 1, 2, 3$). If h were a constant, then Ω would not depend on the x's at all and the y's would all be constants. The solutions of Eq. 5.1 would then be straight lines in space-time—that is, the histories of "particles" moving with constant velocities.

In Hamiltonian language, the space-time curves satisfying Eq. 5.1 are *rays,* and it is at this precise point that that word enters the Hamiltonian theory of water waves. Those who have used what is essentially this method have been a little timid about nomenclature, preferring the word *orthogonal* to *ray,* and indeed the rays are generally, but not always, at right angles to the waves. But *ray* is undoubtedly the proper word, forced on us if we seek a common ray-wave basis for the discussion of all types of waves.

It must be emphasized that a ray is not a curve drawn on the surface of the water; it is a space-time curve, and so represents the history of a moving "particle," quite fictitious if you like. On examining the speed at which these fictitious particles move (this speed is contained in Eqs. 5.1), we find that it is precisely that speed commonly referred to as the group velocity. Now the group velocity, which differs from the phase velocity in a dispersive medium, has long been recognized as the speed at which energy is transmitted. We have words for these fictitious particles: for elastic waves, the *phonon*; for electromagnetic waves, the *photon*; for matter waves, the *electron* (or whatever particle is involved); for gravitational waves, the *graviton.*

Why should the water wave be a Cinderella among waves? If all wave types can be subsumed under one Hamiltonian formulation, then for each type there should be an appropriate name for that fictitious particle which travels with the ray (or group) velocity. In fact, you cannot discuss a thing without calling it by a name, and anyone who delves into the Hamiltonian theory of water waves will recognize this necessity. Accordingly, I venture to suggest the name "hydron" for such fictitious particles associated with water waves. This name was evolved in discussion with W. F. C. Purser, who has been working with me on this Hamiltonian theory and has contributed to the clarification of basic ideas (*5*).

Epilogue

Do hydrons really exist? As part of a mathematical scheme for the discussion of idealized water waves, they certainly do exist, but it is hardly likely that anyone will succeed in catching one in a bottle on the seashore. Nevertheless, the concept and the word may possibly be of some service in discussing how ocean waves carry energy. When a roller breaks on a beach it causes the beach to vibrate and warms it up a little, and we may say without laughter that the energy is carried away by phonons and photons. Is it not pleasant to be able to say that the breaking of a wave is a conversion of hydrons into phonons and photons?

References and Notes

1. W. H. Munk and M. A. Traylor, *J. Geol.* **55**, 1 (1947); "Breakers and Surf, Principles in Forecasting," *U.S. Navy Hydrographic Office Publ. No. 234* (1958); J. B. Keller, *J. Fluid Mechanics* **4**, 607 (1958); for other references, see J. J. Stoker, *Water Waves* (Interscience, New York, 1957), p. 133.
2. See J. L. Synge, in *Handbuch der Physik,* S. Flügge, Ed. (Springer, Berlin, 1960), vol. 3, pt. 1, p. 109, where, however, the approach is oriented toward classical dynamics rather more than is desirable in the present connection.
3. J. L. Synge, *ibid.,* p. 124.
4. Here w is a parameter; it is not the time x_3.
5. W. F. C. Purser and J. L. Synge, *Nature* **194**, 268 (1962).