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The Nature of Mathematics

Both constructive intuition and the study of abstract structures characterize the growth of mathematics.

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Some of the most noted mathematicians and philosophers have addressed themselves to a discussion of the nature of mathematics, and I can hope to add very little to the ideas they have expressed and the insights they have given; but I shall attempt to draw together some of their ideas and to view the issues in the perspective that seems to me appropriate to the present state of mathematical scholarship, taking account of the great increases that have been taking place in the body of mathematical learning, and of the changes in viewpoint toward the old and basic knowledge that grow out of deeper understandings brought about by generations of mathematical research.

In discussions of this subject we find a sharp difference in the views of able mathematicians. This reflects the concern of some that the trend toward abstraction has gone too far, and the insistence of others that this trend is the essence of the great vitality of present-day mathematics. On one thing, however, mathematicians would probably agree: that there are and have been, at least since the time of Euclid, two antithetical forces at work in mathematics. These may be viewed in the great periods of mathematical development, one of them moving in the direction of "constructive invention, of directing and motivating intuition" (*1*), the other adhering to the ideal of precision and rigorous proof that made its appearance in Greek mathematics and has been extensively developed during the 19th and 20th centuries.

The first position, that the emphasis on abstraction has gone too far, is presented by Courant and Robbins in *What Is Mathematics?* though their position is modified by their recognition of the power of the axiomatic method and the deep insights it has made possible. They say, in part (*1*): "A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules and syllogisms, without motivation or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can the free mind achieve results of scientific value."

The second point of view is represented classically by Bertrand Russell's famous definition of mathematics as the "subject in which we do not know what we are talking about or whether what we say is true." Of this view Marshall Stone has this to say (*2*): "A modern mathematician would prefer the positive characterization of his subject as the study of general abstract systems, each one of which is an edifice built of specified abstract elements and structured by the presence of arbitrary but unambiguously specified relations among them." Stone says in two other passages (*3*): "While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world . . . At the same time . . . mathematical systems can often usefully serve as models for portions of reality, thus providing the basis for a theoretical analysis of relations observed in the phenomenal world." "Indeed, it is becoming clearer and clearer every day that mathematics has to be regarded as the corner-stone of all scientific thinking and hence of the intricately articulated technological society we are busily engaged in building."

In the history of mathematics the emphasis in research is sometimes on constructive intuition and the acquisition of results without too much concern for the strict demands of logic, sometimes on the insights gained by the identification and study of abstract systems within a carefully designed logical framework. But over the years the body of mathematics moves forward inevitably with growth in both directions. An individual mathematician chooses to work on one frontier or the other and the emphasis changes from one period to another, but mathematics as a whole and the community of mathematicians have their obligation to the total spectrum. For mathematics is the servant as well as the queen of the sciences, and she weaves a rich fabric of creative theory, which is often inspired by observations in the phenomenal world but is also in-

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spired often by a creative insight that recognizes identical mathematical structures in dissimilar realizations by stripping the realizations of their substance and concerning itself only with undefined objects and the rules governing their relations.

As Von Neumann has said (*4*): "It is a relatively good approximation to truth . . . that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetical motivations."

Euclid and the Parallel Postulate

With this introduction, it will be useful to consider briefly those episodes in the history of mathematics that play a decisive role in the development and understanding of this dichotomy. The Greeks made fundamental contributions in parts of mathematics other than geometry; in addition to Archimedes's wide-ranging interest in applications, I cite only Euclid in number theory and Eudoxus in analysis. But the failure of the Greeks to develop adequate symbols with which to express many of their ideas made their treatment of these subjects cumbersome. Through Euclid's *Elements,* however, they contributed to mathematics the ideal of the development of a body of knowledge proved by logical deduction on the basis of a limited number of axioms, a concept that has exercised enormous influence.

One of the greatest of Euclid's contributions to geometry was his recognition that the parallel postulate could not be derived from the others. For 2000 years after Euclid, the development of geometry is characterized by attempts to prove the parallel postulate. At last, in the time of Gauss at the beginning of the 19th century, the problem was solved. And what a solution! A geometry developed independently in Germany by Gauss, in Hungary by the Bolyais, and in Russia by Lobatchevski in which this postulate does not hold, and in which the sum of the angles of a triangle is less than 180 degrees. Interestingly enough, Gauss's impulse was to check to determine whether our physical world (and here he meant only the earth on which we live) was described by Euclidean or by this new non-Euclidean geometry. He found that his instruments were not good enough to discriminate; but it is of some interest to recall that the non-Euclidean geometry developed later by Riemann, in which the sum of the angles of a triangle is greater than 180 degrees, was found by Einstein to provide a satisfactory framework within which to develop his ideas of the physical universe. In passing, it should be noted that the parallel postulate, unlike the others, deals with lines that cannot be described by finite considerations. Infinity early raised difficulties for mathematicians, and the subsequent development of our subject sees infinity introducing new and exciting vistas, which, however, are recurrently accompanied by logical problems that have caused an upheaval in mathematical thought.

The successful denial of the parallel postulate—the recognition that the assumption of a contradictory postulate could be used as the basis for the description of a consistent geometry, one which in fact proved later to be useful in describing the physical universe—opened up a whole new world to mathematicians. The requirement that axioms be self-evident became meaningless, and in its stead were substituted the requirements of consistency and completeness. Exploration of this new-found freedom in the choice of axioms led to the development of many other kinds of abstract geometry, and, in algebra, there was a veritable feast of new ideas, as new number systems were explored by varying one axiom after another, or by recognizing, after the discovery of new systems, that their essential structure could be described in terms of an axiom system closely related to one that was well known but different from it in one or more of its axioms. The axiomatic method has provided deep insights into mathematics, disclosing identities where none had been suspected. In the hands of mathematicians of genius this method has been used to strip away exterior details that seem to distinguish two subjects and to disclose an identical structure whose properties can be studied once for all and applied to the seperate subjects. Thus, if we consider three familiar ideas—the addition of real numbers, the multiplication of the numbers in a finite number field, and the result of performing in succession two displacements in Euclidean space—and, for all three, study only the skeleton remaining when each is thought of as a set of abstract elements with an appropriate law of combination, we quickly see that each can be described as a group. And properties of the three may be studied together by the axiomatic theory of groups. The nature of the elements is irrelevant to the study of the properties that follow from the axioms.

The group is an example of one of the three basic mathematical structures that we now recognize. It is one kind of so-called "algebraic" structure. The other two basic structures are called "ordered" and "topological," and each can be described abstractly, the first concerning itself with a generalization of the usual "less than or equal to" relation, the second with the notion of continuity. Modern mathematics is increasingly concerned with systems that satisfy at once the axioms for two different kinds of structure. An example of this is given by the complex numbers. When at the beginning of the 19th century the great discovery was made that complex numbers could be represented geometrically in the Euclidean plane (a familiar topological space), all the available insights about the plane could be used to gain familiarity with the nature of complex numbers.

Many systems, such as the complex numbers, can be characterized by a conjunction of the properties of two of the three kinds of basic structure. And there are many contemporary mathematicians who are interested in the study of known mathematical systems in terms of algebras, ordered systems, and topological spaces.

From Euclid to Gauss

In moving into a discussion of the axiomatic method, I omitted any mention of the great eras of mathematical development from the time of Euclid to the time of Gauss. But it was in this intervening period that a domain wide-flung and vastly influential was conquered by mathematicians whose driving force was intuition and construction, who ignored the axiomatic approach of the Greeks and made brilliant leaps on the basis of intuition, analogy, and guesswork. One need only mention the names of Descartes, Fermat, Pascal, Newton, Leibnitz, and Euler to indicate the vast scientific territories that were conquered in the 16th and 17th centuries. Analytic geometry, many facets of analysis and number theory, probability theory, and the calculus were initially developed in these centuries (*5*). And later centuries have seen this kind of mathematical discovery continue and expand. It is of interest that the contemporary French mathematician Hadamard takes the position that "the object of mathematical rigor is to sanction and legitimize the conquests of intuition." As we emphasize the deductive structure of our science and of acceptable proof, let us not lose sight of the fact that many of the most significant results that we prove were arrived at by guesswork, by intuition, by brilliant insight.

Role of the Unsolved Problem

The role of the mathematical conjecture, of the unsolved problem in the development of mathematical ideas, should be pursued further. In periods of great mathematical activity there has always been a lively interchange among mathematicians. The long attempt to prove the parallel postulate and the revolutionary impact of the discovery that it was independent of the others have already been mentioned. Other great problems whose solutions were decisive milestones in the history of mathematics are well known. The early assumption that all ratios of lines are rational was disproved when the Pythagoreans established that the ratio of the diagonal of a square to its side is irrational, or, as we would say, that the square root of 2 is irrational. With this discovery the Pythagoreans introduced some of the basic problems of modern mathematical analysis—the concept of the infinite, of limits and continuity. The pursuit of nonalgebraic irrationals has been carried on for centuries; many aspects of the treatment of the infinite remain unresolved.

Another famous unsolved problem is the one usually referred to as Fermat's last theorem. Actually Fermat, who was a mathematical genius of the 17th century although he was professionally a lawyer and public official, had an intriguing way of announcing his results without stating his full proof, particularly in the theory of numbers. Fermat's last theorem is stated on a margin of his copy of the second book of Diophantus' *Arithmetica*, where he wrote, after noting the solution in integers of the familiar equation $x^2 + y^2 = a^2$, "On the contrary it is impossible to separate a cube into two cubes, a fourth power into fourth powers, or, generally, any power above the second into two powers of the same degree. (In other words, the equation $x^n + y^n = a^n$ has no solution in integers if n is greater than 2). I have discovered a truly marvelous demonstration which this margin is too narrow to contain."

This is the famous last theorem which he stated in 1637. Mathematicians have been at work on this problem ever since that time. The attempts have not been successful, but they have led to important advances in mathematical knowledge. It was his work on this theorem that led Kummer in the 19th century to the introduction of ideals, with the consequent reestablishment for algebraic integers of the fundamental theorem of arithmetic, the theorem that assures the unique factorization of integers into primes, without which our concept of integer sits most uncomfortably. The extension of Kummer's work by Dedekind and Kronecker has been central to the development of modern algebra. Nowadays we are apt to read in the newspaper about the solution of a famous unsolved mathematical problem. For example, the New York *Times* of 27 April 1959 carried an editorial called "The mathematical age" that began: "Mathematicians made news twice last week as the solution of two historic problems was announced at a meeting in this city. For most of us, no doubt, the subjects of these two problems, automorphic finite groups and Latin squares, are rather remote. But we are willing to take the word of professional mathematicians that two important new steps have been taken across the mathematical frontiers."

One of the most famous sets of mathematical problems was formulated by David Hilbert, the eminent German mathematician who died in the 1940's. In his lecture at the International Congress of Mathematicians held in Paris in 1900 he described his now famous problems. Before stating his problems, Hilbert had this to say (*6*): "The great significance of specific problems for the advancement of mathematics in general, and the substantial role that such problems play in the work of the individual mathematician are undeniable. As long as a branch of science has an abundance of problems, it is full of life; the lack of problems indicates atrophy or the cessation of independent development. As with every human enterprise, so mathematical research needs problems. Through the solution of problems, the ability of the researcher is strengthened. He finds new methods and new points of view; he discovers wider and clearer horizens."

Search for Consistency

One of the problems that Hilbert enunciated on this occasion was disposed of in 1931 by Kurt Gödel, now at the Institute for Advanced Study at Princeton. Gödel's paper has been called one of the century's main contributions to science, and something should be said of it. But first, let me put this problem of Hilbert in its setting. The 19th century saw a great surge forward in mathematical research. Gauss, one of the giants of all mathematical history, began to change the whole appearance of mathematics. A fertile intuition, and inspired mathematical inventiveness, combined with a concern for rigor, made Gauss's contributions to mathematics of first importance in all the branches of mathematics studied in his time—in arithmetic or number theory (which he called the Queen of Mathematics), in geometry, in analysis, in algebra. Indeed, Gauss's work is an ornament of the whole of mathematics. In the 19th century mathematics moved on many fronts, but one, in particular, was to introduce problems that have even now not been solved. At the end of the 19th century George Cantor introduced the notion of sets, a powerful new tool which, however, in its 20th-century development has brought with it paradoxes and so-called antinomies that have undermined the confidence of mathematicians in classical logical processes as they affect the infinite. A series of paradoxes produced by the type of reasoning used by Cantor in his theory of infinite sets led to a critical examination of all mathematical reasoning. Whitehead and Russell, at the beginning of the 20th century, tried to show that, by proper methods, we can avoid the set-theoretic contradictions, and that all of mathematics can be derived from logic. In this they failed, but their work has had tremendous influence. At about the same time the intuitionists, of whom the Dutch mathematician Brouwer was a leader, tried to avoid the contradictions introduced by the use of classical logic by insisting that all proofs be constructive, that we avoid the law of the Excluded Middle. This law is the basis for the method of proof, familiar in high school geometry, that begins by assuming that the desired result is not so and shows that this assumption leads to a contradiction. The new methods avoided logical paradoxes, to be sure, but a great portion of the mathematical results that had been found during the preceding centuries could not be proved by the new constructive methods. Hilbert, who had achieved eminence through the astonishing variety of his contributions to many fields of mathematics, including algebra, analytic number theory, analysis, and the foundations of geometry, himself began the search for a rigorous proof of the consistency and completeness of one substantial part of mathematics such as arithmetic. He sought to show that no two theorems deducible from the postulates can be mutually contradictory, and that every theorem of the system is deducible from the postulates. In 1931 Gödel proved that Hilbert's search was hopeless—that it is impossible, within a system broad enough to encompass ordinary arithmetic, ever to prove the consistency of the system in question, and that there is always a proposition of arithmetic which can be formulated within the system that can neither be proved nor disproved by a finite number of logical deductions made in accordance with the procedures of the system.

The hazards in using much of classical mathematics have never been removed. But there are certain results and concepts that mathematicians feel must be kept, either, as R. L. Wilder says (*7*), "for application to physical problems or, at the other extreme, for the building up of mathematical theory itself . . . We find that in order to study the properties [of these concepts]—which is . . . necessary in order to improve their utility as mathematical tools —we have to augment the older methods of proof with new methods. And at this point the old bugaboo of the mathematician rears its ugly head—the fear that the new methods may introduce contradictions. Here is where the mathematical logician gets to work . . . whenever we find that new concepts and methods engender inconsistency, we shall, if the concepts seem to make for progress, try to patch up our methods before we reject the concepts." The late E. H. Moore is quoted as having said, "Sufficient unto the day is the rigor thereof."

Language of the Sciences

Standards of logic change as mathematical research progresses, and we are bound by the standards of our time. It is in the study of the properties of new concepts, in the deeper understanding and mastery of older concepts, in the development of technical facility in handling those that have been solidified into theories that the enrichment of mathematics as the language of the sciences lies. Such understanding and mastery constitute the distinctive contribution that the mathematician brings to the increasingly many fields of physical and social science and engineering in which mathematics is being used, and this mastery must include the ability to recognize a mathematical concept in a concrete situation and trim it of its attributes so that it may be studied with mathematical techniques. For mathematical concepts and techniques, derived solely because of their interest and quite independently of possible use, have repeatedly proved their usefulness. There is, for example, the application of matrix theory to quantum mechanics, of topological results to nonlinear mechanics; there is the use by Einstein in the general theory of relativity of the concepts developed by Riemann in his treatment of non-Euclidean geometry; and there are other instances too numerous to mention. The fact is that there is no field of mathematics clearly marked as the only one appropriate for applications, and it is true that the most unexpected applications of seemingly abstract and remote fields have been found and are being found repeatedly. Moreover, problems arising in the natural and social sciences continue to enrich the fabric of mathematics. Seemingly all mathematics is the language of science. The critical facility, for conversing in this language, is the ability to think of the problem, which is usually presented in many frills like a lady in her Easter finery, in mathematical form, to "construct the mathematical model," as we say. Once the trimmings have been removed, the machinery of mathematics comes into play. This makes it possible to derive mathematical theorems, results that can be translated back into the original natural situation, so that their predictions can be checked against experience. The final test of the suitability of the model is this checking against the real world.

When the same procedure is used to study purely mathematical problems, the jump to the theorems is often made by intuition, by analogy, by guess, without the process of abstraction and model building. In practical problems it is when such an intuitive guess cannot be made by the engineer or physicist that the mathematician is consulted.

And now, as I conclude, let me state the major positions that seem to me to emerge from considerations such as those I have set forth. They are these:

—That mathematics is a language which must be learned and that the arsenal of techniques of mathematics must be mastered if we are to speak this language.

—That mathematics grows by the addition of new theorems, and that the discovery of new theorems is made sometimes by insights furnished by intuition, sometimes by insights provided by abstraction and the identification of patterns.

—That the proofs of theorems rely on the logic of their day, but that mathematicians are constantly concerned to find the logic that makes the proofs of needed theorems adequate.

—That mathematics is both inductive and deductive, needing, like poetry, persons who are creative and have a sense of the beautiful for its surest progress.

—That many of the problems of mathematics come from mathematics itself, but that many more, at least in their earliest genesis, come from the realities of the world in which we live.

—That realms conquered by mathematics solely because of their intrinsic interest to mathematicians have provided in the past, and continue to provide, parts of the conceptual framework in which other scientists view their worlds.

—That the process of abstraction and axiomatization has provided simplification and a deep understanding of the body of mathematical results and a powerful tool for conquering new mathematical worlds.

References and Notes

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4. J. Von Neumann, in *The World of Mathematics*, J. R. Newman, Ed. (Simon and Schuster, New York, 1956), vol. 4, p. 2063.
5. A number of examples, taken from the records of this period, of mathematical results originally conjectured on the basis of observation and daring guess, and subsequently proved, are described by George Polya in his two-volume work *Mathematics and Plausible Reasoning* (Princeton Univ. Press, Princeton, N.J., 1954).
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