Approaches to Mathematics

Should mathematics be considered an ethereal creation, a source of practical applications, or a mental game?

Wallace Givens

Three sharply distinct approaches to mathematical exposition are illustrated in the material considered here; the review is concerned with two books—**Studies in Modern Analysis** edited by R. C. Buck [Prentice-Hall, New York, 1962. viii + 182 pp. $4] and **Mathematics: A Cultural Approach** by Morris Kline [Addison-Wesley, Reading, Mass., 1962. xv + 701 pp. Illus. $7.75]—and with six monographs translated from a Russian series—**Popular Lectures in Mathematics**: vol. 1, *The Method of Mathematical Induction* by I. S. Sominskii (vii + 57 pp.); vol. 2, *Fibonacci Numbers* by N. N. Vorob'ev (viii + 66 pp.); vol. 3, *Some Applications of Mechanics to Mathematics* by V. A. Uspenskii (vii + 58 pp.); vol. 4, *Geometrical Constructions Using Compasses Only* by A. N. Kostovskii (xi + 79 pp.); vol. 5, *The Ruler in Geometrical Constructions* by A. S. Smogorzhevskii (viii + 86 pp.); and vol. 6, *Inequalities* by P. P. Korovkin (vii + 60 pp.) [I. N. Sneddon, Translation Editor. Blaisdell, New York, 1961. Paper, $0.95 each]. Since the levels of difficulty of the subject matter and the audiences addressed differ as markedly as the styles, direct comparison is not relevant. Nevertheless, it is illuminating to contrast the aims, successes, and defects of the three approaches.

The slim and elegant first volume of the Mathematical Association's series, *Studies in Modern Analysis*, contains papers by E. J. McShane, "A theory of limits"; M. H. Stone, "A generalized Weierstrass approximation theorem"; Edgar R. Lorch, "The spectral theorem"; and Casper Goffman, "Preliminaries to functional analysis." (Although the book is distributed by Prentice-Hall, it is volume 1 of the Mathematical Association of America's series, *Studies in Mathematics*, and is available to MAA members for $2, if ordered from MAA.) The general purpose of these studies is "to bring to the general mathematical community expository articles at the collegiate and graduate level on recent developments in mathematics and the teaching of mathematics" in the hope "that these will help to overcome the communication barrier which has arisen as a natural consequence of the tremendous acceleration in mathematical development that has taken place, especially within the last twenty-five years." With so important and difficult a goal, a critical evaluation can properly go beyond the usual report on contents and consider the nature of the "communications barrier" itself.

As R. C. Buck notes in his discerning introduction, these papers present topics that are fundamental for pure and applied analysis and what is more important, "they succeed in the more difficult task of conveying some of the *attitudes* that are characteristic of modern mathematicians." For readers with a good undergraduate background in mathematics, who are willing to make a serious intellectual effort and *who are not repelled by the attitude conveyed,* the book is a splendid success. If, however, the communications barrier consists in some measure of antipathy on the part of those who are interested in applications toward studying the skeletal structure of what they prefer to regard as a full-bodied entity, then the book fails rather completely, since it gives no heed to this reaction.

The Squeeze

To reply that the book is intended for students of mathematics, not for students of applications, does not counter this criticism. The difficulty is that, if the gate to graduate work in mathematics is made too narrow, as it is now, only those with a limited spectrum of interests will go through. If it were possible to gain all the needed knowledge after turning aside, all could be well. But it is not. The *attitude* of modern mathematicians has a profound and, despite all the current bitter controversy, an increasing importance in ever broader areas of human life. To convey this attitude to an adequately large number of readers will require a less uncompromising approach than the one encountered here, however successful it is for a limited audience.

In McShane's article, a modern general theory of limits, stemming from the Moore-Smith theory of 1922, unifies various concepts of limit in the language of families of sets called "directions" and simplifies considerations by extending the real number system ("since infinite limits are too useful to reject") to include plus and minus infinity. An applied mathematician might be defined, for the moment, as one who would object (I think unwisely) that not only is the machinery too heavy for the "obvious" results obtained but that his attention has been diverted from the interesting distinction between convergence and divergence.

Stone's exposition of the Stone-Weierstrass approximation theorem occupies a third of the volume, uses a far broader range of concepts, is more difficult, and deserves Buck's praise when he writes "I cannot think of a better introduction to the spirit of modern mathematics." Here even the applied mathematician should admit that the results justify the machinery used.

Lorch motivates the study of eigenvalues, in an informal style, in three-dimensional space, and carries the discussion somewhat beyond formal statements of the spectral theorem for completely continuous and for bounded transformations in Hilbert space. An extreme case of the fashionable aversion to the display of a matrix is found on page 98 where the simple form of the spectral theorem is interpreted by introducing a basis, constructing a matrix, and distributing the eigenvalues on the diagonal " 'from northwest to southeast.' " Despite this concession to an absurd extreme of fashion, the author's style augurs well for his book on spectral theory (announced for spring 1962 by Oxford University Press).

Goffman gives a clear introduction to functional analysis, which he describes as "that branch of mathematics in which elements of a given class of functions

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are considered to be points of an appropriate infinite-dimensional space." The elegance of the space methods of modern mathematics is apparent in the discussion of fixed-point theorems, compact spaces, Hilbert spaces, and Banach spaces and algebras. The historical supplement and bibliography are interesting and valuable.

Dangerous Culture

Kline's long book, *Mathematics: A Cultural Approach,* contains much culture and some mathematics. In the preface, the author states that the "book is primarily intended for one-year terminal courses addressed to liberal arts students" and that it "should serve in the training of teachers of elementary mathematics of secondary-school teachers in *non-mathematical* subjects" (my italics). It seems particularly important to stress even this slight limitation imposed by the author, since his views, as reported in a column-long article in the New York *Times,* are sharply critical of the introduction into the schools of the new approach to mathematics; this was regarded by many mathematicians (and I am included in this group) as giving aid and comfort to those in the educational system who wish an excuse to ignore an important development that would require of them an unpleasant or impossible intellectual effort.

There is so much in this book which is valuable and which should be known and appreciated by every cultured person that it would be pleasant to ignore the exceedingly serious defects. It would, however, be a calamity if the book were used as the textbook for teacher training, to say nothing of teacher *education,* courses, and a reviewer is obligated to establish this fact. There are two basic reasons why the book, for all its genuine merits, is dangerous.

The first is that 20th century mathematics hardly exists, even by implication, in it. Indeed, although the concepts clustering about the word "set" are central to the clarity of definitions and derivations in mathematics (one must think in terms of "the *set* of all solutions" rather than of "*a* solution" or "*the* solution"), Kline discusses the relatively unimportant algebra of sets, thus missing the whole point of the matter. Since George Boole died in 1864, Kline's inaccurate evaluation (on page 592) of the significance and consequences of his work is nearly a century out of date. If it were true that the mathematics done in this century had the same ratio to that of the preceding two millennia as 60 does to 2000, one could forgive the negligible oversight of ignoring this century. In the mid-thirties Hermann Weyl said that he felt sorry for the next generation of mathematicians, since the great generalizations had been made and now the details must accumulate to support the next great advance. Most mathematicians would think that this pessimism has been shown false, and they would say that mathematics continues from triumph to triumph. Even when we acknowledge that the flood of new results being obtained contains a dangerously increasing amount of trash and that our perspective on recent work is inadequate, we see that great advances of a fundamental sort have been made, and they are hardly hinted at in this book.

The Role of Mathematics

The second major flaw seems to stem from the attitude that if only one avoids an excess of rigor and explains matters clearly, then mathematics is really very simple and can be appreciated intuitively. Indeed (on page vi) Kline writes that the central tenet of the philosophy on which the book is based is that "knowledge is not additive but an organic whole and that mathematics is an inseparable part of that whole." Without taking exception to Kline's admirable effort to inform his readers of the deep roots of mathematics in culture, one must insist with all force that it is the mathematician's almost unique role to abstract from the environment the study of the structural interrelations of abstract concepts which, however much they arise in perceptual or unconscious reality, nevertheless have an *independent* reality in the human intellect. Kline clearly states his disagreement with this view. Thus (again on page vi) he says: "Abstractions taught independently of the totality from which they are abstracted are useless and meaningless." He goes on to say that "there is beauty in mathematics proper, but this is hardly to be found in most of the material that one must teach at the elementary level." If this is true, it is a tragedy that will nullify much of the effort now being expended to improve the school curriculum in mathematics.

Inadequacies of Approach

Chapter 26, "Non-Euclidean geometries and their significance," could well have partially countered my first main objection—the use of the book as a text for anyone. Although non-Euclidean geometries were developed in the early part of the 19th century, the full flowering of the lack of inevitability in mathematical concepts has taken place in our generation, and Kline could have used this chapter to illuminate this. To some extent he does make the attempt, but without conviction.

Two technical points will illustrate the difficulty of Kline's approach. On page 556, in discussing Gauss's work, he mentions the fundamental theorem of algebra in the form: "every algebraic equation of any degree has at least one root." Even worse, on page 64, after an unsatisfactory attempt at explaining that $\sqrt{2} + \sqrt{3}$ is a number since the summands are, he goes on to say, that "when we add 2 and ½ . . . the summands are really not combined." One would wish these were isolated bits of carelessness, but in reading these 700 pages I reluctantly reached the conclusion that Kline does not satisfactorily clarify the concept of number or of a number being a solution of an equation. A second serious technical inadequacy occurs on pages 318 to 320 where, in a section entitled "Functions and formulas," the fundamental concept of function is so loosely described that the terms in the title are hardly distinguished.

Space will not permit printing the dozens of quotations I gleaned from the book, but there are stimulating and thoughtful comments that could well be read by professional scientists who are aware of the book's limitations. The author's style is such that these are discursive rather than incisive. Thus, on page 673, a quotation from Felix Klein, advising mathematicians that their "privilege of freedom implies the obligation of responsibility" could well be printed and hung in the entrance hall of many mathematics departments. Indeed this final chapter, "The nature and values of mathematics," contains much wise comment.

The extent to which Kline accepts the frequently encountered and previously mentioned antipathy toward "bloodless" mathematics (a more accurate term than "pure") is apparent on page 1 where, after a displayed quotation from Bacon unboundedly praising mathematics, the first sentence is:

"One can wisely doubt whether the study of mathematics is worth while and can find good authority to support him." Were it not for the fact that Kline has written a book of "kleine Mathematik" one could take much pleasure in his partially successful attempt to persuade the indifferent and opposed that the classical concepts of mathematics, including algebra, geometry (including projective), trigonometry, and calculus (described almost intuitively) have influenced and remain important in human culture.

These first six volumes in the *Popular Lectures in Mathematics* series (published by Blaisdell Publishing Company but copyrighted by Pergamon Press in 1961) are all translations from Russian originals dated 1951 and 1952 (volumes 2 and 4) or 1957 to 1959. In length they vary from 57 to 79 pages.

Each volume is attractively priced ($0.95), but for this one pays the price of an unattractive page resulting from the photographic reproduction of typescript. Worse, there are a good many places (particularly in volume 6) where superscripts and the like are badly reproduced to the extent that an inexperienced reader will be seriously inconvenienced. The mathematical public deserves to be warned that other instances of such carelessness by this printer (Pergamon) have occurred.

In striking contrast to the appeal to practical importance, as in Kline's book, or to efficiency and economy, as in the MAA study, the six Russian authors explore their subjects "for fun" and with casual disregard of time, space, and practicality. Repetition from varied points of view is deliberately used to secure clarity, and indeed it is said (volume 1, page 12) with the utmost good sense that "In order to learn to use the method of induction it is necessary to discuss a sufficient number of problems." This is not carried to the extreme of repetitive tedium, at least not for the age level addressed. The preparation assumed varies somewhat, but it is noted (typically) in the foreword to volume 1 that for chapter 1 and for part of chapter 2 the first 7 years of school mathematics is sufficient and that the mathematics course of a full secondary school suffices for the remainder. In volume 6 the elements of inequalities are assumed known, and the author notes that "the independent solution of a few difficult problems will undoubtedly be more useful to the pupil than the solving of many simple ones." He continues: "We suggest therefore that the pupil turns to the solutions only after he has found an independent solution, possibly differing (which is very good!) from the solution indicated by the author." (The current flurry of interest in this country in a simple-minded version of teaching machines comes unfavorably to mind.) The manner in which such detailed attention to pedagogical purpose is applied must of course vary with subject matter and audience, but it seems to me to be the key to successful mathematical exposition.

Volumes 1 and 6, *The Method of Mathematical Induction* and *Inequalities,* are of the most general mathematical interest, but volume 2 takes one delightfully and easily from the origin of Fibonacci numbers in the "rabbit problem" of nearly 750 years ago to continued fractions and geometric paradoxes. In volume 6 (on page 15) time has revised the statement that e and π have been calculated to 808 decimal places, since a hundred thousand have now been calculated. Also in volume 6 are interesting problems in maxima and minima, treated simply yet without benefit of the calculus.

Some Applications of Mechanics to Mathematics (note: *not* the reverse application) is fascinating and should stimulate the imagination of some students, but the volume is often so casual that the inclusion of the word "proof" in quotation marks, as on page 12, is well advised and the acknowledgment (on page 56) that "All the reasoning based on mechanical considerations can appear incorrect to fussy readers" is justified.

The two volumes on geometrical constructions are replete with many figures and the style is that of synthetic geometry, so proofs are generally imprecise by modern standards. Since the exposition is careful, this is not inappropriate for student readers. Volume 5 contains, as part 1, "Some theorems of synthetic and projective geometry," which culminate in the theorems of Brianchon and Pascal, and the book's title also serves as the title of part 2.

To summarize: the MAA study is chilled, dry wine, appropriate for a formal dinner by candlelight; the translations are a well-aged vintage from an undistinguished estate, about which one may be comfortably sentimental. Kline's book is, of course, a full case of pop. What is one to read? Forget professional advancement, and even sleep, and read them all: *e pluribus unum.*

Max-Planck-Institute

New Methods of Cell Physiology. Applied to cancer, photosynthesis, and mechanism of x-ray action. Otto Heinrich Warburg. Interscience, New York, 1962. xv + 644 pp. Illus. $34.50.

This beautifully produced volume of publications from the Max-Planck-Institute for Cell Physiology (Berlin) contains reprints of 103 papers completed between 1945 and 1961; 90 papers are written in German, and the remainder in English. About half deal with various aspects of photosynthesis; 18 are concerned with the metabolism of tumor cells, while the others are divided among investigations on the mode of action of x-rays, enzymology, and a number of miscellaneous topics, including methods for the isolation and determination of a number of substances in biological preparations.

Most of the papers reprinted here have previously been published in the literature, and they include work not only by Otto Warburg himself, but also by a number of other people who were, at some time during the period covered by the book, associates of Warburg at the Institute. These reprints encompass reports of experimental work in the various fields, as well as a number of reviews and discussions, such as the well-known paper by Warburg, "On the origin of cancer cells," which first appeared in *Science* [**123**, 309 (1956)].

The hitherto unpublished communications are divided among the various topics in roughly the same proportion as are the other papers in the book. Almost all of the new papers are very short, ranging in length from a few lines to three or four pages. Several deal with extensions and adaptations of manometric techniques: methods are described, for instance, for the determination of ascorbic acid, for the independent generation of three different gases during the course of an experiment without opening the reaction flask, and for the measurement of carbon dioxide partial pressures over bicarbonate-carbonate mixtures. Others describe investigations of the quantum requirement of photosynthesis, the influence of the partial pressure of carbon dioxide on the quantum requirement, the breakdown and resynthesis of glutamic acid in *Chlorella*, and studies on photosynthetic phosphorylation, including the inhibition of this activity by