Mathematical Inutility and the Advance of Science

Should science entice the mathematician from his ivory tower into Solomon's House?

Carl B. Boyer

A few years ago institutions of learning were cutting requirements in mathematics and foreign language, and Phi Beta Kappa, worried about the survival of the liberal arts, took the drastic step of establishing for initiates minimum requirements in language and mathematics. Today the attitude has changed; but if the contemporary return to favor of mathematics results from a panicky concern for defense, the revival may be short-lived. Thus it is that mathematicians find themselves in the equivocal position of endorsing the demands for increased mathematical training at the same time that they look askance at the motives. Training in mathematics is just as appropriate for philosophers and statesmen as for sputnik-builders; but we shall argue here a more modest thesis concerning the role of mathematics in science, raising a voice in protest against two extreme views. One of these was forcefully expressed in 1941 by G. H. Hardy in *A Mathematician's Apology* (*1*): "It is not possible to justify the life of any genuine professional mathematician on the ground of the 'utility' of his work. . . . I have never done anything 'useful.' No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."

The only usefulness he granted mathematics was as an "incomparable anodyne." Hardy went so far as to distinguish between what he called "real" mathematics and "trivial" mathematics —the former being nonuseful, the latter "useful, repulsively ugly and intolerably dull."

Where Hardy rejoiced that the remoteness of mathematics from ordinary human activities keeps it "gentle and clean," Lancelot Hogben, at the other extreme, in 1937 wrote in *Mathematics for the Million* (*2*, p. 36) that "mathematics has advanced when there has been real work for the mathematician to do, and . . . it has stagnated whenever it has become the plaything of a class which is isolated from the common life of mankind." Both of these extreme views do violence to the history of mathematics and science. History indicates on the one hand that the growth of mathematics and the concomitant advance of science are not chiefly the result of utilitarian pressures, but it teaches also that activities of mathematicians which once appeared to be inconsequential have in the end been of far-reaching significance in the growth of science. Paradoxically, the mathematician seems to have been most useful to science when the apparent inutility of his activity was especially marked. Today, especially, surrounded as we are by pressures of immediacy and expediency, it is necessary to look beyond the caricature of the mathematician as a glorified calculator and to appreciate the part that pure mathematics has played in the long-range growth of science.

Pre-Hellenic Mathematics

It was customary, a generation ago, to argue that pre-Hellenic mathematics was entirely practical, but it is obvious now that this picture was overdrawn. Some of the problems in the Ahmes papyrus, for example, are far from utilitarian in nature; and the mathematical inutility in the Egypt of almost four thousand years ago is matched in the Mesopotamian valley of the same period by an instance recently uncovered by Neugebauer. Indefinitely many right triangles with integral sides were known to the Babylonians, for they had the equivalent of a formula for such Pythagorean triads. If p and q are arbitrary integers, with $p > q$, then $p^2 - q^2$, $2pq$, and $p^2 + q^2$ form such a triple of numbers. This result, one of the most remarkable from Old-Babylonian mathematics, is a sophisticated bit of number theory far removed from the hope of immediate utility.

It becomes clearer all the time that mathematical inutility was not unknown in the pre-Hellenic period; but with the Greeks it seems to have become a passion. Greek mathematics started out soberly enough with an eye to the practical. Geometry took its name from the measurement of the earth, and soon it was projected into the heavens; arithmetic promptly found applicability in the Pythagorean discovery that music is "number in motion." But then, probably toward the beginning of the last third of the 5th century B.C., came a discovery which was poles removed from the world of the practical man, and this left a deeper mark on mathematics than has any other single event in its history. Two line segments, it was found, might be such that the ratio of their lengths is not expressible as a ratio of integers. That the diagonal of a square, for example, is incommensurable with its side is of no consequence for the engineer with his slide rule, but in Greece this devastating discovery paved the way for the classical deductive development of mathematics. Ultimately, of course, the deductive method spilled over into the sciences, for it was found to have practical, as well as esthetic, value.

The 5th century B.C. bequeathed also to mathematics the three famous problems of antiquity—the duplication of the cube, the squaring of the circle, and the trisection of the angle—and the better half of later Greek developments centered about these. Inasmuch as craftsmen of the time could solve each of these with a precision that would challenge the keenest senses to find a flaw, the problems made sense only to the impractical geometer, and, as was discov-

The author is a member of the department of mathematics of Brooklyn College, Brooklyn, N.Y.
This article is adapted from a vice-presidential address delivered 29 Dec. 1958 in Washington, D.C., at a joint meeting of the History of Science Society and section L of the AAAS.

ered in modern times, all three of them are, as presented, impossible of solution. Could anything be more futile than to tackle problems which are meaningless to the practitioner and beyond the power of the scholar? History here has amply vindicated the activities of the ivory-tower mathematician, for the search for solutions led to discoveries without which modern science as we know it would have been unthinkable. Conic sections, for example, seem to have been discovered by Menaechmus, tutor of Alexander the Great, in the course of his efforts to duplicate the cube, and although the utility of the ellipse, parabola, and hyperbola escaped Greek scientists, we know that without the speculations of Menaechmus there might have been no laws of Kepler, no law of gravitation, and no lunik.

Eudoxus and the Great Triumvirate

At the Academy of Plato, as among the Pythagoreans, mathematics was a class-related subject far removed from the common life of mankind, and yet the subject flourished exceedingly. The chief contribution of Eudoxus, the outstanding mathematician associated with Plato, was a theory of proportion which is the equivalent of modern definitions of real number, and it is to be doubted that any practical scientist has had occasion to use the principle of Eudoxus or can tell what a real number is. Eudoxus also had a hand in the method of exhaustion, and this was about as impractical a forerunner of the calculus as could be imagined. Nevertheless, without Plato, the "maker of mathematicians," and the work of Eudoxus, the bulk of what we think of as Greek mathematics would not have developed.

The last century of the Hellenic period might be called the "heroic age," for it was then that the characteristically Greek problems and principles were formulated. During the "golden age" which followed, these were elaborated by the great triumvirate of Euclid, Apollonius, and Archimedes. The earlier sections of Euclid's *Elements*—those included in modern elementary textbooks—have a flavor of practicality, but the deeper one goes, the further the material departs from the ordinary world; one finds a proof of the infinity of primes, a formula for perfect numbers, and the crowning Platonic theorem that there are but five regular solids. In the *Conics* of Apollonius are elaborated the properties of curves, which at the time were beautiful and impractical, for the ellipses which we see in the heavens, the hyperbolas which are formed by our lamp shades, and the parabolas we descry in our suspension bridges were not there for the Greeks. Even the quadratures of Archimedes, which anticipated the now indispensable integral calculus, had at the time little utility; and Archimedes' most sophisticated treatise, *On Spirals,* was largely a mental exercise in circle-squaring and angle-trisecting.

Sharp Decline

Conflicting conjectures have been advanced to account for the sharp decline in mathematics following the great triumvirate, but there is general agreement on one aspect—an admitted transfer of interest from pure to applied mathematics. Under the practicalist theory the shift in interest to the popular fields of astronomy and mensurational geography should have been a catalyst for rapid mathematical development, not the herald of centuries of doldrums. Let this be a warning to those who would equate mathematics and measurement, or who would espouse the fragile thesis of Tobias Dantzig, in *Number, the Language of Science* (1930), quoted (with approval) by Hogben (*2*, p. vii): "It is a remarkable fact that the mathematical inventions which have proved to be the most accessible to the masses are also those which exercised the greatest influence on the development of pure mathematics."

I have mentioned above the mathematical inventions of greatest influence in the pure mathematics of the Greeks, and these inventions were neither accessible nor of interest to the masses. There was in ancient Greece another type of mathematics which had wide appeal. Computation and arithmetic methods, stemming from Babylonian views, were what concerned the vast majority—not axiomatics—and the place of Heron and Diophantus becomes clearer when one regards them as representatives of a tradition which always was present in Greece but which shows through only rarely because of the loss of ancient works. Occasionally both traditions—the higher axiomatic or nonutilitarian stream and the lower arithmetic or utilitarian current—appear in one and the same individual. Ptolemy's *Almagest,* for example, is akin to classical geometry, while his astrological *Tetrabiblos* adopts the Babylonian arithmetical devices, and the verdict of history has been that the theoretical *Almagest* was more influential in the advance of science than the pragmatical *Tetrabiblos.*

No better illustration of the baneful effect of the cold breath of utility upon the ardor of the mathematician can be found than in ancient Rome, where the consequence for science of the Roman contempt for mathematical inutility is too well known to require repetition here. Let us hope that history will not repeat itself in this respect and that a tough-minded concern today for the immediate and obvious needs of national defense—just such as the Romans had in mind—may not stifle the legitimate interests of the pure mathematician. Administrative agencies in this country (and apparently in Russia also) thus far have been very far-seeing in this respect and have generously supported basic research, but if the public clamor for more mathematics in the schools were to result merely in fostering development of expedient techniques, the results could be tragic indeed.

The consequences of a lack of interest in the principles of mathematics, as distinct from a concern with practical outcomes, can be seen in the medieval civilizations—Latin, Greek, Chinese, Hindu, and Arabic. Not one of them had a vigorous tradition of pure mathematics and, interestingly enough, none was strong in science. Much has been made of the so-called Hindu-Arabic system of numeration, but even granted that it was an invention of the Hindus (which is not definitively established), it should be noted that the system involved no principles not known in antiquity, and that with it the Hindus and Arabs were able to do but little. Only later, in 16th-century Europe, was a significant mathematical advance made.

The Renaissance and Mathematics

A facile explanation of the opening of the new age sometimes is found in the rise of a merchant class with practical computational needs, or in the explorations which posed geographical problems, or in the establishment of closer relations between the scholar and the artisan, but the revival in mathematics does not fit neatly into any of these. Apart from the recovery of the Greek treatises in pure geometry, the event which marked the opening of a new era was the publication of the algebraic solution of the cubic equation. On the surface this looks like an eminently practical result, but nothing could be more deceptive. The formula which Del Ferro and Tartaglia discovered and which Cardan published in 1545, just two years after the epoch-making treatises of Copernicus and Vesalius, was not then, and is not now, of use to the applied mathematician or the practicing scientist. It gave a strong fillip to the pure mathematician's pursuit of algebra, but it did not satisfy the practitioner's need for a practical device for getting approximations to the roots.

Nevertheless, the radical solution of the cubic did in the end stimulate the advance of science—indirectly, and in a rather curious way which well illustrates the unexpected role that mathematical inutility plays. The new formula called attention to imaginary numbers, for in some mysterious way they were bound up with the real roots in the so-called irreducible case. Cardan said of the arithmetic in this case that it is "as subtle as it is useless," and Bombelli, his contemporary, described it as "a wild thought, in the judgment of many; and I too was for a long time of the same opinion." Today any electrical engineer can attest to the ultimate utility of such useless wild thoughts on imaginary numbers; but these numbers at first were rejected by practical men, and even by some not generally regarded as excessively utilitarian. Of them Simon Stevin wrote, "There are enough legitimate things to work on without need to get busy on incertain matter"; and only occasionally were men bold enough to handle these quantities which Leibniz regarded as a sort of amphibian, halfway between existence and nonexistence.

Contemporary with Stevin was François Viète, an inadequately appreciated mathematician who likewise valued mathematical inutility. Trigonometry in its infancy had been so unfortunate as to be immediately applicable to astronomy and navigation, and hence, as a science of indirect measurement, it had had a limited growth. By subordinating the practical art of solving triangles to the liberal study of relationships among the trigonometric functions, Viète did much to convert the subject into a branch of pure mathematics, sometimes known as goniometry, or analytical trigonometry. Today in secondary schools the solution of triangles is giving way to increased emphasis upon the analytic side of trigonometry, and every electrical engineer, every student of optics and acoustics, knows through the work of Viète that the immediately practical is not in the end necessarily the most useful.

Descartes, Fermat, and Boyle

It is in the 17th century that one expects to see the other side of the coin—aspects of mathematics which were suggested by experience and which directly promoted the advance of science. Much of this there was, but less, I suspect, than is commonly assumed. Analytic geometry, for example, was not the practical outgrowth of a mundane use of coordinates. Descartes regarded his geometry as a triumph of philosophical method to be appreciated by the elite, and it took form in his mind as a generalization of an impractical locus problem inherited from ancient Greece. Apollonius had considered the locus of points for which the product of the distances to two of four given lines should be proportional to the product of the distances to the other two lines. Pappus had suggested, but was unable to complete, the generalization of this to six, eight, ten, or more lines, hinting at a geometry of more than three dimensions—the height of inutility, one should suppose. About this problem Descartes developed his coordinate geometry, the aim of which at the time was the theoretical geometric construction of the roots of equations that now would be solved by the practical man through successive arithmetical approximations.

Fermat, an independent inventor of analytic geometry, represents an even more striking instance of mathematical inutility, for he was as unconcerned about the practical outcome of his studies as he was about personal fame. And yet Fermat was an inventor in three branches which turned out to be among the most useful of all: he discovered the fundamental principle of analytic geometry; he invented the differential calculus; and he was a founder of the theory of probability. His coordinate geometry was scarcely more practical than Descartes'. It was a study of geometric loci, the "crowning point" of which was the following proposition: Given any number of fixed lines, the locus of a point from which the sum of the squares of the segments drawn from the point to meet the lines at given angles is constant is a solid locus (conic section).

Can this be used in the workaday world? His new infinitesimal analysis did turn out to have tremendous practical implications, but Fermat's thought here, too, was nonutilitarian. Perhaps the best way to describe his calculus is to say that it represented the first satisfactory definition of the tangent to a curve, a bit of theory which Newton and Leibniz developed into an algorithm which made possible the celestial mechanics upon which our hopes for space travel are founded. Even Fermat's theory of numbers, at the time far removed from the market place, has not been entirely without applicability, for his studies in figurate numbers enter into statistics.

Francis Bacon, in his utopian Solomon's House, had valued mathematics solely for its utility, but Robert Boyle, Fermat's Baconian contemporary, put in a good word for mathematical inutility. Boyle realized with regret that in mathematics one cannot in old age atone for the sins of neglect in one's youth, and it was not lack of training in practical mathematics that he regretted. "I confess," he wrote (*3*), "that after I began . . . to discern how useful mathematicks may be made to physicks, I have often wished that I had employed about the *speculative* part of geometry, and the cultivation of the *specious* Algebra . . . a good part of that time and industry, that I had spent about surveying and fortification . . . and other parts of *practick* mathematics" (italics mine).

The *Principia* of Newton probably never would have been written had it not been for the work of Fermat and others like him, and hence it can be regarded as the fruit of earlier mathematical inutility rather than as an inevitable outgrowth from social and economic roots of the time. In fact, there is not so large a proportion of applied mathematics in the book as is commonly supposed. Moreover, the philosophical import of the law of gravitation far transcended any practical significance. It should be noted also that Newton's contribution in this connection was not so much a discovery—some half a dozen men earlier had suggested an inverse square law—as it was a mathematical proof of the validity of the law, and the practical man has no truck with mathematical demonstration. Newton derived as a corollary of the law of gravitation the fact that within the earth the force varies directly as the distance from the center—a bit of knowledge which at the time served no useful end but which carried within it the germs of potential theory and paved the way for the electromagnetic age.

In Time of Crisis

It is in times of crisis akin to our own that the temptation to undervalue mathematical inutility is great, but mathematicians of stature generally have risen above this. Few more striking instances of this can be found than during the French Revolution. Lazare Carnot and Gaspard Monge were key figures in the frantic defense against foreign invasion, yet during the turmoil they did not yield to the exigencies of the moment and divert their efforts to applied mathematics alone. Both men spent much time reviving pure geometry, one of the more beautiful but less immediately useful branches, and their names still are associated with theorems in the subject. Carnot, the "Organizer of Victory," wrote an especially useless work—one on the metaphysics of the calculus, which has gone through many editions down to our time—and Monge was instrumental in the establishment of the École Polytechnique, an institution which might well be taken as a model of balance between pure and applied mathematics.

Lagrange, one of the teachers at the school, spent much of his time looking for a logical foundation for the calculus —a pursuit which scientists of the time regarded as misdirected effort, but which has since led to the theory of functions, a subject which physicists find indispensable. But the theory of functions owed even more to what at the time looked like a fruitless effort. During the Napoleonic era no less than three men were toying with the idea of picturing imaginary numbers, and the result, now known as the Argand or Wessel or Gaussian diagram, became the basis for the theory of functions of a complex variable, with striking consequences for science. It probably is not too much to say that electrodynamics is the gift of the imaginary number, once shunned as useless.

Nineteenth Century Developments

Most ages have produced men who studied mathematics with little regard for its applicability, but the 19th century was a veritable paradise of mathematical inutility. One of the amazing things about this penchant of the century is that it proceeded in the main from *anciens élèves* of the École Polytechnique, a school of technology. In France, pupils of Monge stirred a revival in pure geometry such as had not been seen since the days of Apollonius. Projective geometry, with its concern for ideal elements, and the analytic geometry of imaginary points fascinated the heirs of the French Revolution, inapplicable though these studies might be. Poncelet, an engineer in the French army under Napoleon, reached the epitome of mathematical inutility when he noted that all circles in a given plane have two points in common—not ordinary points, of course, but two points which are both imaginary and at infinity! The two chief mathematical journals of the time both carried in the title the phrase (one in French, the other in German) "Pure and Applied Mathematics," but so obvious was the preponderance of pure mathematics that wags read the title as "Pure Unapplied Mathematics." And treatises of the time showed the same tendency.

The imaginary appeared everywhere in analysis, geometry, and algebra, and especially in the works of Cauchy. And what was the effect upon science of this feast of uselessness? It probably is safe to say that physics, at least, never developed more rapidly than during and immediately following the period we have been describing. Mechanics, optics, thermotics, acoustics felt the effect of Cauchy's theory of functions of a complex variable. But how, one may be inclined to ask, can the theory of the imaginary number have anything to do with the real world? The answer, of course, is that imaginary numbers are not fictitious, despite their name. What one generation labels impossible, another reduces to common sense. After Gauss, Wessel, and Argand had shown that imaginary numbers can be pictured as points in a plane, it was a short step to Sir William Rowan Hamilton's identification of the theory of complex numbers with the properties of couples of real numbers. This led Hamilton to devise a four-dimensional analog—the system of quaternions—and this in turn was later generalized into the theory of tensors, without which the mathematical theory of relativity would be unthinkable.

Relativity is in a real sense a bequest to science of once-useless mathematics. Not only is it an outcome of the imaginary number; it resulted also from some impossible geometries. Gauss, greatest mathematician of all times, played indifferently with useful and useless mathematics. His contributions in probability and statistics found ready application; much of his theory of numbers, which he enjoyed most, still is without palpable use. Among the mathematical toys of Gauss was one called non-Euclidean geometry, of which similar schemes were developed independently by Bolyai and Lobachevski.

The new geometries seemed to be a denial of common sense, but disagreement with sense never has been, and we hope never will be, a bar to mathematical investigation. If it had been, the 19th century would not have pursued the study of geometries of more than three dimensions. As it turned out, both non-Euclidean geometry and multidimensional geometry are applicable to science in the theory of relativity. Bertrand Russell has said that Riemann is logically the immediate predecessor of Einstein, and one might add that Cayley's geometry of n-dimensions, developed in 1843 with no inkling of possible applicability, has since found a place in thermodynamics, applied chemistry, and statistical mechanics. Here in America the analysis of Gibbs once was termed a "hermaphrodite monster," but the monster soon was tamed and became the chemist's best friend. Perhaps even today's bizarre mathematics of transfinite numbers eventually may become a scientist's man Friday. Had Hamilton been dissuaded on utilitarian grounds from toying with economically worthless noncommutative algebras, much of the abstract algebra of the 20th century would never have developed, and quantum mechanics would have been the loser.

The history of science seems indeed to support the findings of psychology in the thesis of a great nonagenarian mathematician of our day, Jacques Hadamard, who, on the basis of a study of *The Psychology of Invention in the Mathematical Field* (4), concluded that "practical application is found by not looking for it, and one can say that the whole of civilization rests on that principle."

References

1. G. H. Hardy, *A Mathematician's Apology* (1941), pp. 59–60, 90.
2. L. Hogben, *Mathematics for the Million* (1937), p. 36.
3. R. Boyle, *Works* (London, 1772), vol. 3, p. 426.
4. J. Hadamard, *The Psychology of Invention in the Mathematical Field* (1945), p. 124.