Mass of the Moon from Satellite Observations

Abstract. The four classical methods of determining the mass of the moon are noted, and a new use of an artificial earth satellite is proposed. The procedure, based on Kepler's law, is outlined, but at present the uncertainties in the observed data preclude an improved estimate of the lunar mass.

One of the more difficult astronometric problems is the calculation of the mass of the moon (*1*). This particular measurement has been performed by making geocentric observations of the position of the sun, planets, or the asteroid Eros. The monthly to-and-fro oscillations in the precise position of these objects are caused by the motion of the earth around the center of gravity of the earth-moon system. This displacement and the known distance of the moon are easily related to their relative masses.

A second method is based on a determination of the mass at the distance of the moon sufficient to cause the lunar component of the tides. A third method is based on a term in the expression for the motion of the equinoxes. The moon's attraction on the equatorial bulge causes a torque, or couple, about an equatorial axis, which exhibits itself in a monthly noding of the poles (the nutation). A similar torque, due to the sun, produces the equinoctial precession. The known distance of the sun and moon allows a comparison of the mass of the moon to that of the sun.

A fourth method, based on the exact form of Kepler's law (Eq. 1) requires the precise measurement of the lunar parallax and the period and gives the sum of the masses of the earth and moon.

With a satellite of very small mass we may take the ratio of the conditions for dynamic stability (Kepler's law),

$$G(M_e + M_m) = 4\pi^2 \frac{a_m^3}{T_m^2}$$

and (1)

$$G(M_e + M_s) = 4\pi^2 \frac{a_s^3}{T_s^2}$$

where M_e, M_m, and M_s are the masses of the earth, moon, and satellite, respectively, and a_m, a_s, T_m, and T_s are the mean distances and periods of the moon and satellite, and rearrange, neglecting the mass of the satellite,

$$\frac{M_m}{M_e} = \left(\frac{a_m}{a_s}\right)^3 \left(\frac{T_s}{T_m}\right)^2 - 1 \qquad (2)$$

Obviously, the data used to compute the mass ratio must be observational. Note that, astronomically, the term *mean distance* does not mean the average distance, but is the semimajor axis of the undisturbed elliptic motion. Furthermore, the period is not the observed time of circumscription, since the period of a close body about an ellipsoid depends upon its inclination and orbital eccentricity as well as its major axis dimension (*2*).

The effect of the accuracy of the observed artificial satellite data on the value of the lunar mass obtained must be examined. Differentiating Eq. 2, we obtain

$$\frac{\delta(M_m/M_e)}{(M_m/M_e) + 1} = \frac{2\delta T_s}{T_s} - \frac{3\delta a_s}{a_s} \qquad (3)$$

From Eq. 3 it is evident that the error depends upon the absolute magnitude of the ratio and therefore a small uncertainty in the measurement produces a large uncertainty in the ratio. The present accuracy of artificial satellite measurements is insufficient to warrant the presentation of such calculations. However, the observations are capable of such precision, and the value of the lunar mass determination may be improved over the present 5 percent uncertainty.

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References and Notes

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A pH Calculator Based on Linear Transformations of the Henderson-Hasselbalch Equation

Titration curves of weak acids and bases are usually represented by symmetrical sigmoid curves based on the Henderson-Hasselbalch equation. The equilibrium of hydrogen ions, neutral molecules, and the ionized form in buffer solutions is governed by the law of mass action. The logarithmic form of the relation is

$$pH = pK + \log (B/A)$$

where pK is the negative logarithm of K, the equilibrium constant; B refers to the base, defined as the proton-acceptor form of the weak electrolyte; and A refers to the acid, the proton-donor form.

The equilibrium of the two forms with hydrogen ions is expressed by the relation

$$A = B + H^+$$

For example, in the buffer system acetic acid–sodium acetate, acetate ion is the conjugate base, while the undissociated molecule is the acid. In the system ammonium chloride–ammonium hydroxide, ammonium ion is the acid and ammonium hydroxide the base.

In either of these systems, if the acid is titrated with sodium hydroxide, the titration curve is calculated from the formula

$$pH = pK + \log [b/(a - b)]$$

where a denotes the equivalents of acid initially present and b is the number of equivalents of added base. If, on the other hand, a strong acid, HCl, is used in the titration, the curve is represented by

$$pH = pK + \log [(b - a)/a]$$

where b is constant. In either case the curve is symmetrically sigmoid when pH is plotted against added base, b, or acid, a. The point of inflection is at the midpoint, where pH is equal to pK, and B is equal to A. For any series of univalent weak acids and bases, a family of titration curves exists in which all the curves are similar. The position of each curve with respect to the pH axis is determined by pK.

If the negative logarithms of A and B are denoted respectively by pA and pB, the Henderson-Hasselbalch equation may be expressed in the form

$$pH - pK = pA - pB$$

In this form of the expression the titration curves for any series of weak univalent acids and bases become linear and identical. At the midpoint of the line, pA and pB are equal, and pH is equal to pK. Transformation to the linear form has the advantage of eliminating the calculations necessary to construct the sigmoid form. Accurate construction of the sigmoid curve requires the calculation of four or more points on either side of the midpoint, a total of eight or ten points. For the linear form it is necessary only to draw a straight line on semilogarith-

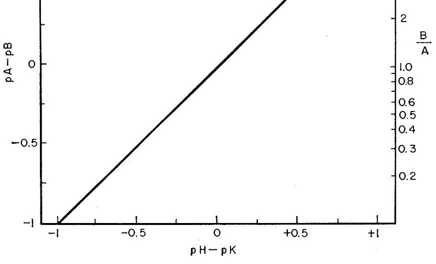

Fig. 1. Linear transformation of Henderson-Hasselbalch equation. Ordinate: $(pA - pB)$; abscissa: $(pH - pK)$. On parallel ordinate values of the ratio B/A are given on logarithmic scale.