nitrogen ratio cannot be considered as constant. (iv) An atmospheric model must be founded on heat transport such as suggested by Chapman. (v) It is possible to estimate temperatures and temperature gradients in the neighborhood of 500 km and, consequently, conservative values of densities, if various values of the heat conduction are adopted.

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New Approach To Teaching Intermediate Mathematics

It is based on a resolution of the spectra of meanings of the letter x and the term *variable.*

Karl Menger

Nothing is more distasteful to an active mathematician or scientist than discussions of symbolism and notation, and that dislike is perfectly understandable. After having overcome in his youth whatever difficulties the formal expression of ideas presents, the mathematician finds that certain ways of writing have become his second nature and regards any suggestion of a change, even if he recognizes its merits, as nothing but a trivial nuisance.

There are, however, situations in which a thorough discussion of such matters on the highest level is inevitable. They occur when, at turning points in the history of culture, it becomes imperative to make certain techniques and ideas of mathematics available to wider strata of the population. In the large groups to be initiated, many persons lack the ability to overcome the difficulties that the specialist overcame in his youth. Moreover, an immense collective benefit results if even persons with that ability are spared unnecessary complications.

Dr. Menger is professor of mathematics at Illinois Institute of Technology, Chicago.

Such a turning point affected arithmetic when, during the Renaissance, mercantilism and experimental science were born. In banks and laboratories, the letters introduced by the Greeks and Romans as numerals proved to be utterly inefficient, even though they had served arithmeticians for over 2000 years. Unfortunately, medieval mathematicians misinterpreted the specialists' manipulative facility as intrinsic simplicity of the ancient numerals and regarded the Hindu-Arabic ideas as a pure nuisance. "Even in the 15th century," wrote G. Sarton, "there were still any number of learned doctors and professors who claimed that the Roman letters were much simpler than the Hindu numerals." Such prejudices confined the knowledge of arithmetic to a small elite and retarded its democratization as well as its progress. Eventually, however, as everyone knows, practical exigencies prevailed —incidentally, to the ultimate benefit of pure mathematics too.

The middle of the 20th century appears to be another such turning point. This is the time when scientific and technological progress has reached proportions necessitating the dissemination, on a large scale, of intermediate mathematics. A considerable part of the population should learn certain techniques of algebra, analytic geometry, and calculus, as well as some basic ideas of those theories. The attempts toward this aim, which follow traditional lines, are generally regarded as not sufficiently successful. In my opinion, the principal stumbling block is the fact that most of those great mathematical ideas and techniques are being presented in their 17th century form.

Uses of x

A principal feature of those antiquated formulations is the indiscriminate use of the letter x (as well as of the letter y) in diverse meanings and according to discrepant rules. What enhances the confusion are references to those diverse types of x and y by one and the same term, namely, *variables.*

Algebra. In algebra, beginners learn that in the formula $x+1=1+x$ they may replace x with numerals, thereby obtaining formulas such as $4+1=1+4$. But they find that this practice must not be applied to the statement that the function $x+1$ is nonconstant, since replacement therein of the "variable" with 4 would lead to the false statement that the function $4+1$ is nonconstant. Beginners further learn that squaring the equation $y=x^4$ yields $y^2=x^8$. But they find that the square of what often is referred to as the function $y=x^4$ is the function $y=x^8$. This contradiction is so blatant that many mathematicians altogether refrain from referring to the said functions as $y=x^4$ and $y=x^8$, and rather call them, briefly, the functions x^4 and x^8. As a result, however, x frequently has various meanings in one and the same statement. An example is the following sentence:

(S_1) The function x assumes the value x for any number x.

This statement (S_1) is being inculcated into beginners until many of them get used to it. Getting used to it means realizing that the function x does not assume itself as a value (even though it assumes the value x), that it is not any number (even though x stands for any number), and so on. In other words, accepting statements of the type of S_1 means realizing that they must not be taken literally. Therefore, there are also many beginners who give up. They have heard so much about the perfect precision of the language of mathematics that, after failing to understand some mathematical statements when taken literally, they regard themselves as lacking any mathematical ability. This group includes persons of great intelligence.

Analytic geometry. In analytic geometry, the beginner learns that a certain parabola is the locus of all points (x,y) such that $y=x^2$ or (which is the same) of all points (a,b) such that $b=a^2$. For instance, this parabola includes the point (3,9) but not the point (9,3). Indeed, $9=3^2$ but $3 \neq 9^2$; in other words, if $a=3$ and $b=9$ then $b=a^2$, whereas if $a=9$ and $b=3$ then $b \neq a^2$. Of course, that parabola may also be said to be the locus of all points (b,a) such that $a=b^2$. This locus, too, includes (3,9) but not (9,3) since $b=3$ and $a=9$ imply $a=b^2$ whereas $b=9$ and $a=3$ do not. All this is in no way surprising. But ask the following question: May this same parabola also be described as the locus of all points (y,x) such that $x=y^2$? Of course, the answer is again affirmative. For instance, this locus includes (3,9) but not (9,3), since $y=3$ and $x=9$ imply $x=y^2$ whereas $y=9$ and $x=3$ do not. Yet this affirmative answer would utterly bewilder the beginner. He would be unable to reconcile it with the fact (likewise taught in analytic geometry) that the parabolas $y=x^2$ and $x=y^2$ are altogether different. The teacher's only hope is that the question mentioned will not be raised and, therefore, that the apparent contradiction will remain unnoticed. For within the classical frame of concepts it is impossible to explain that paradox, the explanation being that mathematicians traditionally use the same pair of letters x, y in discrepant meanings when talking about the parabola $y=x^2$ and about the locus of all (x,y) such that $y=x^2$. That in the latter case one may interchange x and y while in the former one must not is a mere symptom of conceptual differences which traditionally remain inarticulate.

Calculus. In calculus, the reciprocity of differentiation and integration—the very core of the theory—traditionally is expressed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\,\mathrm{d}x = f(x) \qquad (1)$$

for any continuous function $f(x)$ and any number a. Serious shortcomings of formula 1 become apparent in manipulating symbols, even though it has been frequently claimed that the classical symbolism (while perhaps obscuring some of the contents) certainly facilitates manipulative use. In its five occurrences in formula 1, the letter x follows altogether discrepant rules. It may, without any change of the meaning, be replaced with any other letter in its last two occurrences on the left side of formula 1 or in its other three occurrences. For instance, the formulas

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_a^u f(x)\,\mathrm{d}x = f(u)$$

have exactly the same meaning as formula 1. In contrast, formulas resulting from other replacements, such as

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_a^x f(x)\,\mathrm{d}u = f(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_a^u f(x)\,\mathrm{d}u = f(u)$$

are, in general, incorrect. The letter a in formula 1 may be replaced with a numeral as in

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x f(x)\,\mathrm{d}x = f(x). \qquad (2)$$

No such replacement of x in one or more of its occurrences yields a valid formula. For instance,

$$\frac{\mathrm{d}}{\mathrm{d}4}\int_1^4 f(4)\,\mathrm{d}4 = f(4)$$

is utterly nonsensical, while

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^4 f(x)\,\mathrm{d}x = f(4)$$

is, in general, false. He who wishes to state that the two functions equated in formula 2 assume equal values for $x=4$ may replace x with 4 only in its last occurrence. What he must write is:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x f(x)\,\mathrm{d}x\right)_{x=4} = f(4). \qquad (3)$$

In view of such examples, one may well question the traditional claim that the classical presentation is conducive to a purely mechanical handling of symbols.

Needless to say, these complications do not present the slightest difficulties to anyone who has mastered a traditional course in calculus. The reason for mentioning them is the present vital interest in *increasing* the number of people who master the ideas and manipulative techniques of calculus (though not necessarily in their 17tth century form), reminiscent of the Renaissance interest in increasing the number of men able to perform multiplications and divisions (though not necessarily in Roman numerals). True, the time-honored formulation (formula 1) of the Reciprocity Law, which goes back to Leibniz, has been successful during the three centuries that witnessed the activities of Euler, Gauss, and Poincaré. But the Greek numerals were successful during the eight centuries from Pythagoras to Archimedes to Diophantos.

Questions

In view of the quoted (and countless other) examples from intermediate mathematics, what is remarkable is how well teachers succeed in transmitting to many students a feeling for what is right in manipulating x and y and an instinct for which type of variable is present where. (Since the underlying distinctions do not attain the level of the articulate, feelings and instincts are all that can be transmitted.) That, notwithstanding all the teachers' efforts, many beginners, including talented students, give up, is not surprising. Clearly, various circumstances contribute to the unfortunate and dangerous situation in our current mathematical education. But the antiquated symbolic and conceptual frame in which mathematics is being presented certainly is in itself a sufficient reason.

The question naturally arises why these difficulties should come to a head just in this country and at this time. One obvious reason is our attempt to initiate a much higher percentage of the population into intermediate mathematics than do the countries in Central and Western Europe. But there are psychological, in addition to sociological, factors which I carefully studied when, after teaching in various parts of the European continent west of Russia, I initiated thousands of American undergraduates into intermediate mathematics during the war and the G.I. period, and taught hundreds of adults in a metropolitan night school. From these vantage points, for the past 15 years, I have collected the questions that beginners actually ask and have noticed that the same questions are asked again by mature men who did not receive satisfactory answers in their youth—a fact on which they blame the superficiality of their mathematical knowledge. Almost all of those questions concern the symbolic and conceptual frame of intermediate mathematics. They are raised more frequently on this side of the Atlantic because the American youngster approaches the subject with pure common sense and utterly rejects dictatorial solutions. On the other hand, he less enjoys intellectual acrobatics and easily gives up when dissatisfied. Within the traditional frame, unfortunately, most of those questions are simply unanswerable.

New Presentation

It has been in response to these questions that, for the past 15 years, I have developed a new presentation of intermediate mathematics, outlined in numerous papers (*1*) and elaborated in two textbooks (*2*), which I have tried out in teaching hundreds of students. The essence of this presentation is an approach to mathematics that is based on common sense, which thereby furthers the understanding of the material, and which, moreover, results in truly mechanical manipulations.

1) Emphasis is laid on the possibility of talking about mathematical objects and their interrelations somewhat as one talks about people and their family relations, as in the sentence: The father of the paternal grandfather of a person is the paternal grandfather of the father of that person. The words "a person" delimit the scope of the assertion and make clear that it is being proposed about all persons and not only, say, about all living men, or, on the other hand, about all mammals. If *f*, *g*, and = were generally accepted symbols for the words "the father of," "the paternal grandfather of," and "is the same as," respectively, then one might write:

$$fg\mathrm{X} = gf\mathrm{X} \text{ for any person X.}$$

But the letter X in the formula is not self-explanatory and does not delimit its scope. This is why the formula, in order to render the sentence, must be amplified by the explanatory legend "for any person X." Similarly, the square of the fourth power of a real number equals the fourth power of the square of that number. The words "a real number" delimit the scope of the assertion and make it clear that it is being claimed for all real numbers and not only, say, for all integers, or, on the other hand, for all in some way generalized numbers. Using universally accepted symbols and a letter in lieu of the words "a real number," one may write

$$(\mathrm{x}^4)^2 = (\mathrm{x}^2)^4 \text{ for any real number x.}$$

Since the letter x is not self-explanatory, the mere formula would again leave uncertainty about the scope of the assertion, wherefore it must be amplified by a legend. Even the very intent of a mere formula including a letter is in need of clarification. One and the same formula occurs, for instance, in the *assertion* that $\mathrm{x}^2 - 9 = 0$ for x being 3 or -3, and in the problem: find x such that $\mathrm{x}^2 - 9 = 0$. In case of an imperative legend, no assertion is intended, and the letter is referred to as the *unknown* of the problem, whereas the letter in a formula accompanied by a description of its scope is called a *variable,* more specifically, a *numerical variable.* This is the only sense in which the latter term is used in the new presentation. If the writer of a formula that includes letters not designating specific mathematical objects fails to append a legend explaining the intent and the scope of the formula, then he forces his reader to do mere guessing—a procedure strictly shunned in the new approach.

2) The area (say, in square feet) of a square is the second power of the length (in feet) of the side of that square or, in a formula following these words:

$$a(\mathrm{Q}) = s^2(\mathrm{Q}) \text{ for any square Q.} \quad (4)$$

Here, Q serves as what might be called a square variable. In contrast, *a* and *s* designate definite mathematical objects of the type that Newton called *fluents,* namely, area and length in the realm of squares, each fluent resulting from the association of a number with an object of a certain kind. Traditionally, formula 4 is abbreviated to the formula $a = s^2$ connecting the two fluents themselves rather than their values for any square—a situation unfortunately obscured by referring to the fluents *a* and *s* as *variables,* and thereby adding another meaning to that highly equivocal term. Naturally, *a* and *s* must not, in formula 4, be replaced with letters designating any two other fluents (say, perimeter and diagonal), nor should they be interchanged. While $a = s^2$ is true, $s = a^2$ is false, just as, in the realm of numbers, $\mathrm{e} < \pi$ is true and $\pi < \mathrm{e}$ is false. Contrast $a = s^2$ with a statement about many numbers; for example, for any two positive numbers, a and s,

$$\text{if } \sqrt{\mathrm{a}} = \mathrm{s}, \text{ then } \mathrm{a} = \mathrm{s}^2.$$

Here, a and s do not designate fluents. Here, they serve as numerical variables and may, without any change of the meaning, be replaced (for example, with x and y) or even interchanged: for any two positive numbers a and s,

$$\text{if } \sqrt{\mathrm{s}} = \mathrm{a}, \text{ then } \mathrm{s} = \mathrm{a}^2.$$

In the clarified presentation, the conceptual difference between numerical variables and fluents is visibly reflected in a typographical distinction that the reader will note in paragraphs 1 to 5 of this section. Letters in roman type serve as numerical variables, while fluents and functions are designated by *italic* type. This device not only greatly facilitates intelligent reading of mathematics but forestalls a great deal of otherwise almost inevitable confusion. The class of all points (that is, pairs of numbers) (x,y) such that $\mathrm{y} = \mathrm{x}^2$ is the same as the class of the pairs (a,b) such that $\mathrm{b} = \mathrm{a}^2$ or of the pairs (y,x) such that $\mathrm{x} = \mathrm{y}^2$. On the other hand, the parabola $y = x^2$, that is, the class of all points P such that $y(\mathrm{P}) = x^2(\mathrm{P})$, is of course different from the parabola $x = y^2$. Here, *x* and *y* are fluents, the abscissa and the ordinate whose values for the point P are $x(\mathrm{P})$ and $y(\mathrm{P})$. None of the paradoxes of the 17th century notation has to be explained, because in the new presentation none of them ever arises.

3) The (traditionally symbol-less) identity function, which for any number x assumes the value x, is a mathematical object of paramount importance, and it clearly deserves a permanent symbol. If *j* is used as its designation, then the obscure statement S_1 is replaced by:

j is the function such that, for any number x, the value $j(\mathrm{x})$ equals x.

Every statement made in the course of the new presentation may be—in fact, must be—taken literally. The function $j+1$ is nonconstant, and $(j^4)^2 = j^8$. Here again it might be argued that since the lack of a symbol for the identity function has not impaired the success of analysis for the past 300 years, such a symbol must be superfluous. But, at about A.D. 500, Greek mathematicians could say that the lack of a symbol for zero had not impaired the success of their arithmetic for over 800 years. Yet the introduction of the Hindu cipher 0 made arithmetic even more successful and greatly furthered the development of algebra.

4) The distinction between numerical variables, fluents, and functions entails the distinction in calculus between the derivative of a function and the rate of change of one fluent with respect to another fluent—two terms traditionally considered as synonymous even though the derivative associates a function with a function, and the rate of change associates a fluent with two fluents. For instance, the derivative of the sine function is the cosine function (the symbols for the sine and cosine functions are italicized, whereas x serves as a numerical variable):

$$\mathbf{D}\ \mathit{sin} = \mathit{cos}, \quad \text{or } \mathbf{D}\ \mathit{sin}\ \mathrm{x} = \mathit{cos}\ \mathrm{x} \qquad (5)$$

for any x. The rate of change of the distance traveled with respect to the time elapsed is velocity:

$$\mathbf{d}s/\mathbf{d}t = v.$$

For a harmonic oscillator,

$$\text{if } s = \mathit{sin}\ t, \text{ then } \mathbf{d}s/\mathbf{d}t = \mathit{cos}\ t. \qquad (6)$$

Here s and t are specific fluents in contrast to the numerical variable x in formula 5. If s and t in formula 6 are misused as numerical variables, say, by replacing t with π, and s with 0, the result is an implication whose antecedent $(0 = \mathit{sin}\ \pi)$ is valid, while its consequent, $\mathrm{d}0/\mathrm{d}\pi = \mathit{cos}\ \pi$, is utter nonsense. The derivative of a function is its rate of change with respect to the identity function: $\mathbf{D}f = \mathbf{d}f/\mathbf{d}j$, for any differentiable function f. The situation in integral calculus is analogous.

5) In the new presentation, symbols for operations and operators are introduced with great care, avoiding synonyms and equivocations, and in a way that is free of confusing ballast. For instance, the integral beginning at 1 of the function f might be denoted by $\mathbf{S}_1 f$, following the verbal pattern. This symbol bears a relation to the synonymous traditional symbols

$$\int_1^x f(x)\,\mathrm{d}x \text{ and } \int_1^x f(t)\,\mathrm{d}t$$

somewhat like that of "1984" to "MCMLXXXIV" and "MDCCCCLXXXIV." The introduction of any symbol is accompanied by articulate rules concerning its use. In particular, clear stipulations are made as to which part of a formula where that symbol appears is within its reach. One of them is the stipulation that within the reach of an operator symbol (all of which are printed in bold face, as **D** and $\mathbf{S}_1$) is only the immediately following function. On this basis it is clear that $\mathbf{D}\ \mathit{sin}\ \pi$ is the value that the function $\mathbf{D}\ \mathit{sin}$ assumes for π, and not the derivative of the (constant) function $\mathit{sin}\ \pi$, which would be denoted by $\mathbf{D}\ (\mathit{sin}\ \pi)$. Such rules, in conjunction with the use of a symbol for the identity function, make it possible actually to achieve what the classical treatment claims to achieve: to *manipulate formulas in a purely mechanical way.* In the traditional transition from formula 2 to formula 3, like letters in various occurrences are treated in altogether unlike ways. In contrast, the streamlined version of formula 2—that is,

$$\mathbf{D}\ \mathbf{S}_1 f = f \text{ for any continuous } f \qquad (2')$$

—implies $\mathbf{D}\ \mathbf{S}_1 f\mathrm{x} = f\mathrm{x}$ for any x; in particular,

$$\mathbf{D}\ \mathbf{S}_1 f4 = f4. \qquad (3')$$

The transition from general statements to specific formulas proceeds by systematic substitutions and by replacements of variables with designations of specific objects. This technique results not only in a simplification of pure as well as applied analysis but in their complete standardization and automatization.

Conclusion

It goes without saying that, when initiating students into the great ideas of 17th century mathematics in any reformed presentation, one must not neglect to teach them to read the various classical notations, especially those going back to Leibniz, to Lagrange, and to Cauchy. (Our Renaissance ancestors, when disseminating the ideas of arithmetic in the reformed symbolism, taught their students also to read Roman numerals, which in fact are still being taught.) It has been my experience that this aim can be achieved without difficulty. Larger scale experiments would undoubtedly result in further improvements.

The main problem clearly lies in the instruction of *teachers* who have been brought up to regard as nonexistent just those points that cause the beginners' crucial difficulties, and who themselves have never been provided with answers to their students' basic questions—the questions concerning the antiquated frame of intermediate mathematics—even though many teachers feel that those questions are justified. In other words, the problem is *to instruct those teachers to use symbols and basic concepts consistently and to transmit to their students the clarified techniques according to articulate rules.* Considering the remarkable, if partial, success of teachers along traditional lines, one may be confident that, equipped with adequate conceptual and symbolic tools, they will make intermediate mathematics available to such wider strata of the population as the present age demands (*3*).

References and Notes

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