Scientific Work of J. von Neumann

Even before the present age of specialization, few people have ever contributed significantly to several branches of science, and all of them have a permanent record in the annals of the history of science. John von Neumann made fundamental contributions to mathematics, physics, and economics. Furthermore, his contributions are not disjoined and separate remarks in these fields but arise from a common point of view. Mathematics was always closest to his heart, and it is the science to which he contributed most fundamentally.

John von Neumann was born in Budapest on 28 December 1903. He studied in Berlin, Zürich, and Budapest, receiving his doctor's degree in 1926. After serving as Privatdozent in Berlin and Hamburg, he was invited to Princeton University in 1930. Following 3 years there, he became professor of mathematics at the Institute for Advanced Study, a position which he held for the rest of his life. In 1955 he was appointed to the U.S. Atomic Energy Commission and served brilliantly in this post until his death on 8 February 1957.

The earliest significant mathematical work of von Neumann concerns mathematical logic, in which he was a forerunner of the epochal work of Gödel. His accomplishments can be summarized under two headings: axiomatics of set theory and Hilbert's proof theory. In both of these subjects he obtained results of cardinal importance.

Von Neumann was the first to set up an axiomatic system of set theory satisfying the following two conditions: (i) it allows the development of the theory of the *whole* series of cardinal numbers; (ii) its axioms are finite in number and expressible in the lower calculus of functions. Moreover, in deriving the theorems on sets from his axioms, he gave the first satisfactory formulation and derivation of definition by transfinite induction. Von Neumann's work on this subject already showed his power and the elegance of much of his later work. It contained a full clarification of the significance of the axioms with regard to the elimination of the paradoxes. He first showed how the paradoxes are related to the theory of types and then proved that a set exists (this implies that it does not lead to contradictions) if, and only if, the multitude of its elements is not of the same cardinality as the multitude of all things. He also demonstrated that this proposition implies the axiom of choice.

With regard to Hilbert's proof theory, von Neumann clarified the concept of a formal system to a considerable extent. His articles contain the first unobjectionable proof for the fundamental theory that the classical calculus of propositions and quantifiers as applied to computable functions and predicates is consistent.

The work of von Neumann which will be remembered longest concerns the theory of the Hilbert space and of operators in that space. His papers on this subject can be divided into three groups: (i) the properties and structure of Hilbert spaces as such; (ii) studies of linear operators involving in essence only a single operator; (iii) studies of whole algebras of operators.

Von Neumann gave the first axiomatic treatment of Hilbert space and described the relation of Hilbert spaces to all other Banach spaces. A good exposition of his point of view on linear spaces is given in his book on functional operators.

In a remarkable paper, von Neumann gave the complete theory of extensions of Hermitian operators H on Hilbert space to maximal and self-adjoint operators, by means of the Cayley transform $(H+iI)\ (H-iI)^{-1}$. By the same transform, he established the spectral theorem for self-adjoint operators; that is, he constructed a set of projection operators $E(\lambda)$ with the property that H (where $H=H^*$) admits a spectral resolution $(Hf,\ g)=\int\lambda d(E(\lambda)f,\ g)$. He derived a similar theorem for normal operators. The spectral theorem has enormous importance in applications, and von Neumann's work has been of great influence.

Partly in collaboration with Murray, von Neumann founded the theory of weakly closed, self-adjoint algebras ("rings") of bounded linear operators. They first studied "factors"—that is, rings generalizing simple algebras—and developed a "direct sum" theory for rings of operators. The effect of von Neumann's work here is enormous. A whole school has grown up in the past decade devoted to a study of operator rings and their abstract analogs.

In pursuing his researches on rings of operators he was led to introduce the notion of a dimension function into ring theory and found thereby "geometries without points." He developed this theory into his important continuous geometry, which was the subject of his 1937 colloquium lectures to the American Mathematical Society.

The influence of von Neumann's interest in groups can be detected in all phases of his work on operators. In particular, the direct sum theory has many applications in the theory of unitary representations of non-Abelian noncompact groups, as is shown in the work of Mackey, Godemont, Mautner, Segal, and Gel'fand and his school. Von Neumann's work on unbounded operators has heavily colored analysis in the past 25 years. It seems safe to predict that his work on operator rings will color it even more strongly during the next 25.

His contribution to the theory of groups did not stop here. He was the first to show that every subgroup of a matrix group is a Lie group. This result is fundamental to the present techniques for analyzing locally compact groups. He also showed that every compact group can be approximated by Lie groups, and as a consequence that every compact locally Euclidean group is a Lie group. His work on almost periodic functions on groups won for him the Bôcher prize.

His elegant proof of the ergodic theorem stands as one of his important results. Its ramifications have had a profound influence on the study of dynamical systems.

Von Neumann was one of the founders of the theory of games. In spite of the nearly 30 years that have passed since von Neumann's first paper was written on this subject, and in spite of the intensive development of the theory in these 30 years, there is very little in his first paper that would be revised today. It is, as are many of his early papers, strongly under the influence of the axiomatic thinking and gives a formal system which permits a complete description of all the intricacies of a game, with play and counterplay, chance and deception. The paper contains a rigorous definition for the concepts of pure strategy (a complete plan, formulated prior to the contest, making all necessary decisions in advance) and of mixed strategy (the use of a chance device to pick the strategy for each contest). Although similar concepts were used before (by Zermelo and by Borel), no one had used them before with the same incisiveness as von Neumann did when he established the "minimax theorem" for zero sum two-person games. This theorem, which proved valuable for von Neumann's studies in economics, also gave the key to the analysis of games with more than two players, permitting the formation of alliances and camps between the players.

The book, *The Theory of Games and Economic Behavior,* by him and Morgenstern, has affected decisively the entire subject of operations research. Indeed, it may well be said that the present-day importance of the subject results from the influence of this monumental work.

The preceding three subjects are the ones which come to mind at once when writing about von Neumann's contributions to mathematics. However, they are surely not the only fields which have profited from his fertile imagination. He has made significant contributions literally to every branch of mathematics, with the exception of topology and number theory. His knowledge of mathematics was almost encyclopedic—again excepting the afore-mentioned two fields—and he gave help and advice on many subjects to collaborators and casual visitors, possibly to a greater extent than any other present-day mathematician.

It would be very difficult to tell which of von Neumann's contributions to theoretical physics was the more important: the direct or the indirect ones. Four of his direct contributions are known to all physicists. His recognition that vectors in Hilbert space are the proper mathematical concept to describe the states of physical systems in quantum mechanics is unique in the sense that no other person would have realized this fact for many years. Closely related to this observation is his description of quantum mechanics itself. The sketch of his ideas in this connection, presented in chapter VI of his *Mathematische Grundlagen der Quantenmechanik,* still constitutes inspiring reading. Von Neumann's third main contribution is the application of the concept of the mixture of quantum mechanical states—which he invented independently of Landau—to problems of thermodynamics and statistical mechanics. The considerations on irreversibility, in both classical and quantum physics, were his fourth major contribution. These contributions, and some others of a more specialized nature, would have secured him a distinguished position in present-day theoretical physics quite independently of his indirect contributions.

Von Neumann developed several mathematical concepts and theorems which became important for the theoretical physicist; he probably developed them with these applications in mind. In fact, it often seems to the theoretical physicist that the best of von Neumann's mathematical work was motivated by its projected usefulness in some applied science. From the point of view of the theoretical physicist, his two most important mathematical contributions were the theory of nonbounded self-adjoint or normal operators in Hilbert space and the decomposition of representations of noncompact groups, carried out in collaboration with Mautner (both of these are described in the preceding section). Many of von Neumann's colleagues think that his late work, centered around the development of fast computing machines, was also motivated by his desire to give a helping hand to his colleagues in mathematics' sister sciences.

No appraisal of von Neumann's contributions to theoretical physics would be complete without a mention of the guidance and help which he so freely gave to his friends and acquaintances, both contemporary and younger than himself. There are well-known theoretical physicists who believe that they have learned more from von Neumann in personal conversations than from any of their colleagues. They value what they have learned from him in the way of mathematical theorems, but they value even more highly what they have learned from him in methods of thinking and ways of mathematical argument.

Von Neumann's contributions to economics were based on his theory of games and also on his model of an expanding economy. The theory of games has relevance in many fields outside of economics; it answers a desire first voiced by Leibnitz but not before fulfilled. It has been stated (by Copeland of Michigan) that his theory may be "one of the major scientific contributions of the first half of the 20th century." The theory rests on von Neumann's minimax theorem, whose significance and depth are only gradually becoming clear. The theory gives a new foundation to economics and bases economic theory on much weaker, far less restrictive assumptions than was the case thus far. The current analogy between economics and mechanics has been replaced by a new one with games of strategy. Entirely new mathematical tools were invented by von Neumann to cope with the new conceptual situations found. This work has given rise to the publication by many authors of several books and several hundred articles. His study of an expanding economy is the first proof that an economic system with a uniform rate of expansion can exist and that the rate of expansion would have to equal the rate of interest. This study has deeply influenced many other scholars and will unquestionably become even more significant now that problems of growth are being so widely investigated by economists.

The principal interest of von Neumann in his later years was in the possibilities and theory of the computing machine. He contributed to the development of computing machines in three ways. First of all, he recognized the importance of computing machines for mathematics, physics, economics, and many problems of industrial and military nature. Second, he translated his realization of the significance of computing machines into active sponsorship of a computer—called JOHNIAC by his affectionate collaborators—which served as a model for several of the most important computers in the United States. Third, he was one of the authors of a series of papers which gave a theory of the logical organization and functioning of a computer which reminds one of the axiomatic formulation of mathematics, a subject to which he devoted so much of his early youth. In these papers is also formulated a quite complete theory of coding and programming for machines. Here is the complete notion of flow-diagrams and the genesis of all modern programming techniques. In one of these papers is given the criteria and desiderata for modern electronic computing machines.

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