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SCIENCE

Theory of Elementary Particles

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It is obvious that at the present state of our knowledge it would be hopeless to try to find the correct theory of the elementary particles. On the other hand, one may try to form some kind of picture of how such a future theory of elementary particles will look, because, even if we realize that we know only very few details about the elementary particles, we have already quite a good qualitative picture of them, and we feel that even if the experiments go on for 5 or 10 or even more years, this qualitative picture will scarcely change.

Perhaps the best way to start this subject is to give a short review of what we know about the elementary particles, and then the problem of the theory will not be to find the correct theory, but rather it will be to find a model of such a theory. That is, one can, even at the present time, make the attempt to construct theories that at least qualitatively give something very similar to the elementary particles that we see now in nature. Only at a much later stage can we hope to find the correct theory. What I shall try to tell here is in some ways quite ambitious, because it is a model for the real theory of elementary particles—that is, a theory that comprises all knowledge about atomic events, in one single mathematical scheme. On the other hand, it is not so very ambitious, because it is *not* an attempt to find such a theory but only to find a theory that qualitatively resembles it—in other words, a kind of model of such a theory.

Our Knowledge of Elementary Particles

What do we know about the elementary particles? First of all, we know that there is a great number of different elementary particles. We know a mass spectrum of such particles and the masses of many. For instance, the mass of the proton is 1836 times larger than the mass of the electron, so the electron seems to be an especially light particle. Most other particles seem to be heavier by at least a factor of 100. If we consider these masses as something similar to the stationary states in the hydrogen atom, then we see that some of these masses are stable states, and others are unstable states.

The electron apparently is a stable particle and has, therefore, a very sharply defined mass. The proton also seems to be a stable particle, but the neutron is not stable. The neutron can decay, emitting a proton and electron and neutrino. The neutron has a lifetime of roughly a quarter of an hour. Then there are the mesons, which are still much more unstable. Their lifetime is very much shorter. The μ meson has a lifetime of 2×10^{-6} second. The π meson has a lifetime of 2.5×10^{-8} second. The neutral π has a lifetime of only 10^{-15} second. So we see that all different degrees of stability may occur, and as a rule one can assume that when the particles get heavier and heavier, then the chances that they are stable are smaller and smaller, so that probably above a certain mass value all particles will have only an extremely short lifetime. Therefore they will not have a well-defined mass, and then it is of no use to speak about elementary particles.

We know still more about the particles, or, I should say, about the results of any future theory of elementary particles. For instance, we know that any such future theory of elementary particles must contain some invariance properties. It must obey, for instance, all the properties of invariance that are involved in the Lorentz transformation. So they must be invariant for what one calls the inhomogeneous Lorentz group. These invariance properties will lead to a number of conservation laws—conservation of energy, momentum, angular momentum. In connection with quantization, they will also lead to the fact that the angular momentum always is either an integer multiple of $\hbar$ or a half quantum integral of $\hbar$. So all these results must come out of such a theory of elementary particles.

For experiments, these conservation rules mean, for example, that we have selection rules that some particles can only decay into certain other ones. And sometimes we even do not know yet what the selection rules are that apparently are present. For instance, according to all known conservation laws, we think that a proton could disintegrate into a positron and one or several light quanta. But we see that this is not the case; so there must be new conservation laws and, therefore, new invariance properties that have not been accounted for in the present theories.

If we take all this qualitative knowledge together, it seems reasonable to believe that even in 5 or 10 years from now the general picture of this knowledge will probably not have been changed. In 5 or 10 years from now we will certainly know a number of new particles beyond those that we know already. We will have better knowledge of the cross sections and of the production probability of these particles. We will know in what number these particles are created in high-energy collisions, and so forth. But still qualitatively this picture will not have been changed. One important feature of this picture also is that all these particles are connected. By *connected* I mean that when we have a sufficient amount of energy at our disposal—when, for instance, two elementary particles collide at very high energies—then apparently any other type of particle can be created, either directly in the collision or some time after the collision through radioactive decay. So we cannot divide

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all existing elementary particles into different groups that have nothing to do with one another. Such a division is in principle certainly not possible. All the elementary particles are connected.

A Wave Equation for Matter

Let us take this qualitative picture of matter, of the behavior of matter, and ask: How can the theory of elementary particles possibly look? We can say: Since we must have in this theory the invariance for the inhomogeneous Lorentz group, it is very natural that such a theory will in some way be connected with a wave function depending on x, y, z, and t. Because, if we write the wave equation for such a wave function, then it is easy to do it in such a way that the invariance for the inhomogeneous Lorentz group actually is present. This wave equation that we want to write, however, will certainly not be a wave equation for a special kind of waves—light waves or meson waves—or a wave equation for nucleons, or anything like that, because the mesons, light quanta, and nucleons must come out of the equation; they cannot be put into it. So this wave equation, if it exists at all, will be an equation for matter, not for any special kind of elementary particles.

What kind of wave function do we have to introduce to represent matter? We may think, just because we have no other mathematical tools, of functions that are scalars, or spinors, or vectors, or tensors—in any case some of these relativistic functions or operators. It would certainly not be convenient or sensible to start by assuming that this wave function of matter is a scalar or a vector, because then this wave equation could never lead to spinor particles. On the other hand, if we assume that this wave function of matter is a spinor, then there is a chance to represent not only the spinor particles but also the scalar and the vector particles, because, if we start with half integers for spin quantum numbers, we can also get spins that are integers by taking several of these half quanta together; but if we start with integral spin numbers, we can never get the half spin quantum numbers. So it looks natural to assume that, if someday we can write a wave equation for matter, this should be a spinor equation.

Then again, one could think of a spinor equation that is just a linear wave equation, like the Dirac equation. This, however, could certainly not represent the facts, because we know that all elementary particles interact. A linear wave equation, however, will never lead to any interaction, and therefore one cannot expect a linear equation to represent the experimental situation for the elementary particles. So we have to start with a nonlinear equation for a spinor wave function, and we shall see whether we can in this way get a model for a theory of the elementary particles.

What is the simplest nonlinear wave equation for a spinor wave? I think I can quickly write it.

$$\gamma_\mu \partial\psi/\partial x_\mu + l^2\psi(\psi^+\psi) = 0 \qquad (1)$$

Here, $\psi^+_{\text{Heisenberg}} = \overline{\psi}_{\text{Schwinger}} = \psi^\dagger\beta = -i\psi^+_{\text{Pauli}}$, where $\psi^\dagger$ is the hermitian conjugate of ψ and β is the Dirac matrix operator for $(1 - v^2/c^2)^{1/2}$. One can argue that other equations are just as simple or perhaps slightly simpler, but essentially this is a very simple equation. As I said before, I do not believe that this is necessarily the correct wave equation. I just want to see whether such a wave equation can lead to a picture of the elementary particles, which at least qualitatively represents what we know about them.

This wave equation has two parts. The first part is just part of the ordinary Dirac wave equation for a spinor function, $\gamma_\mu \partial\psi/\partial x_\mu = 0$. That would be a Dirac equation for neutrinos. Then there is added a term where l represents a constant of the dimension of a length and $l^2\psi(\psi^+\psi)$ is an interaction term. It is the simplest interaction term one can write. It must be a term of the third order, because with a spinor function it would not be possible to have a second-order term with the correct transformation properties. One could imagine other terms of the fifth order and the seventh order, but this seems to be the simplest one. Also, instead of this term of the third order, one could take other terms with some γ operators in them, but this would not essentially change the situation.

I think qualitatively such an equation seems to be a reasonable starting point for a theory of matter. The question is: Is there any chance that the quantization of such an equation will lead to an ensemble of elementary particles, some stable, others unstable, from which one can then calculate other interactions, and so forth? The next question is: Can such an equation be quantized according to the methods that we know for the quantization of wave fields?

The answer to this latter question is: No, because we know now from the theories of Schwinger and Tomonaga, Feynman, and others that in the quantization of fields, one will always run into the so-called "divergency difficulties," and this can be overcome only in some cases by a formalism, which is called the process of renormalization. Not all equations can be renormalized. On the contrary, we can divide all possible interactions into two types: one type can be renormalized and shows what can be called weak interaction; the other type has what we may call strong interaction, and for strong interactions this process of renormalization does not work. This interaction here, however, belongs to the strong-interaction type, and regardless of what kind of nonlinear wave equation we would write for spinor waves, we would always get the strong-interaction type, which cannot be renormalized. Therefore, we have to invent a new scheme of quantization. We have to change the rules of quantization in such a way that on one side we still preserve those features of quantum theory which we know must be true and still avoid the divergence difficulties and get to mathematical schemes that really work.

Commutation Relationships

The next and most difficult problem in connection with such a wave equation is the question: What assumptions can we make about the commutation relationships? So, we will now be interested in a commutator between ψ at one point and ψ or ψ^+ at another point. Let me write this commutator.

$$\{\psi_\alpha(x); \psi_\nu^+(x')\}_+ = -iS_{\alpha\nu}(x, x') \qquad (2)$$

This commutator is, in this case, written with a + sign between the two expressions on the left side, because we expect for a spinor wave the anticommutation rules that we know from Fermi statistics. The sum $\psi_\alpha(x)\psi_\nu^+(x') + \psi_\nu^+(x')\psi_\alpha(x)$ is, in the ordinary theory, 0 for any nonvanishing spacelike distances between x and x' and becomes a delta function when the points are close together. This anticommutator (multiplied by i) is usually called the S-function, after Schwinger who made much use of it. The problem is: Can we for this nonlinear theory define a new S-function which in a linear theory would be the Schwinger function?

Let me first state some of its general properties. In the linear theory we know that the anticommutator must be 0 whenever the distance between x and x' is a spacelike distance. This is a necessary condition if we want to preserve the properties of causality that follow from the theory of special relativity. From the theory of special relativity, we learn that all action can be propagated only with a velocity less than or equal to the velocity of light. This means that when two points in a four-dimensional world have a spacelike distance, then no action can go from one point to the other, and vice versa. Therefore, at two such points the wave function must always commute, or in this scheme anticommute, because otherwise it would mean that we would have a deviation from ordinary causality. Therefore, we can form the picture shown in Fig. 1. By "0" we indicate the regions where the anticommutator shall be 0. It shall be different from 0 in what one calls the future cone and the cone of the past. The dividing lines between this future cone and those parts where the commutator is 0 form the so-called "light cone." These are the points to which a wave can be propagated with the velocity of light.

Such functions as S in the linear theory are called propagation functions, because they really represent only waves that obey the normal wave equation and are propagated as perturbations from a certain point. There is a singular point $x=x'$, $t=t'$, and from this point a wave propagates into the future or into the past. The function that represents the anticommutator is just such a propagation function. This is so in a linear theory. This is quite understandable, because in a linear theory the commutator itself must obey the wave equation. In a nonlinear case, however, this is not true, and we have first to find the connection between the "propagator" on the one side and the commutator on the other side. Now we have to invent some kind of mathematical trick to see the connection between the propagation functions and the commutator. To find this mathematical connection I will have to write a few formulas. Consider the equation

$$\chi_\alpha(x, x') = \exp\{-i[a_\nu\psi_\nu^+(x') + \text{conj}]\} \psi_\alpha(x) \exp\{i[a_\mu\psi_\mu^+(x') + \text{conj}]\} \quad (3)$$

Right in the middle we find the operator $\psi_\alpha(x)$. Of course, our ψ's are not only functions now, they are also operators, and they shall be noncommuting quantities. And this ψ_α is multiplied on the left side and on the right side with certain factors, which are each other's reciprocal.

The quantities a_ν or a_μ appearing in the exponents shall be the components of an arbitrary spinor with the property of anticommuting with all the wave functions ψ_α. This a_ν is just introduced as a mathematical tool to get the right connection between the propagation functions and the commutator. The factors on both sides of $\psi_\alpha(x)$ depend on x' but not on x. What we have introduced is nothing but a canonical transformation of $\psi_\alpha(x)$ independent of x or α, and

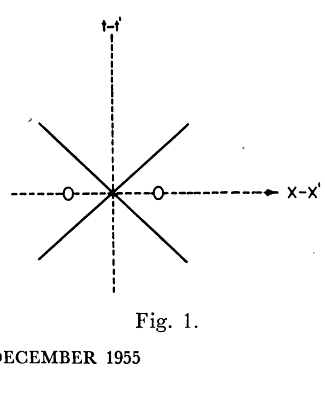

Fig. 1.

therefore one can easily see that the new function $\chi_\alpha(x, x')$ also obeys the wave equation

$$\gamma_\mu\partial\chi/\partial x_\mu + l^2\chi(\chi^+\chi) = 0 \quad (1a)$$

Now we can study what the χ will be like. If we assume that this arbitrary spinor a_ν we have introduced is a very small quantity—and since it is arbitrary we can *take* it as very small—then the expansion with respect to a_ν is the following:

$$\chi_\alpha(x, x') = \psi_\alpha(x) - ia_\nu \{\psi_\nu^+(x'); \psi_\alpha(x)\}_+ + \ldots \quad (4)$$

The first nonlinear expression appearing in the expansion of χ with respect to a_ν is just the anticommutator. And now we can see the relationship between commutator and propagation function. We see that χ apparently corresponds to a solution of the wave equation, which is not a smooth solution but is a solution where superimposed on a smooth solution there is such a perturbation as we have seen from our picture (Fig. 1)—that is, a perturbation that starts from a point $x=x'$. So the χ is a kind of propagation function. It represents a solution of the wave equation that has a perturbation starting from one point. If we assume that the a_ν is very small, then it is a very small perturbation at the point $x=x'$. And the commutator is then the difference between the original smooth solution and this perturbed solution. So now we know qualitatively at least what the connection between commutator, on the one side, and propagation function, on the other side, must be. The commutator must correspond to the difference between two solutions of the original wave equation, one of which is smooth and the other has this perturbation at the point $x=x'$.

Knowing this we can go further and write our function χ_α, the operator that we have defined, in the following way. We can say

$$\chi_\alpha(x, x') = \chi_\alpha^0(x, x') + c_\alpha(x-x') \quad (5)$$

and this definition is to be understood in the following way. The singularity of the function χ_α on the light cone will be contained completely in the c-number function $c_\alpha(x-x')$ (more correctly, the products of the amplitudes a_ν and a c-number). Such a division is actually possible in any present-day quantum theory, because in present-day quantum theory we always assume that the anticommutator at the origin, at the point $t=t'$, is a c-number. For instance, we usually write the anticommutator as a delta function. Here we assume that we do not know which kind of c-number function $c_\alpha(x-x')$ is, but it is an ordinary function and not a field operator creating or annihilating particles. Therefore we may say that we split the χ_α up into one part χ_α^0 which is an operator, but which is smooth at the light cone, and another part c_α which contains the main singularities and is a c-number.

As I said before, this splitting into two parts has always been possible in present-day quantum theory and will, of course, in a linear theory lead to the ordinary commutation relationships. For instance, we can assume that $c_\alpha(x-x')$ is just the vacuum expectation value of the operator $\pm\chi_\alpha$. Then we can put the χ into the wave equation and, if we have done so, we can take the vacuum expectation value of the wave equation, and we find the wave equation for c. Actually it turns out that this function c is a solution of the original wave equation with only slight modifications. Let me write it:

$$\gamma_\mu\partial c/\partial x_\mu + l^2c(c^+c) + c\kappa(s) = 0; \qquad s = \Sigma_\mu(x_\mu - x_\mu')^2 \quad (6)$$

For the c the wave equation that one gets is just the same as the ordinary wave equation that is on the left side and is the one from which we started; however, there is one term added, which is the function c times a function of the space-time distance $(x-x')^2$. Since this last term will not affect the behavior of the c near the light cone, and we are interested only in the behavior of c near the light cone, we can just as well assume that this is approximately constant and say c times a certain constant κ, which we can adjust according to what is convenient in the equation.

This gives us a qualitative picture of how the commutator will look near the light cone and how we can derive these properties from a solution of the wave equation. I should mention the following point before I go on. What we need for all further discussion is just the behavior of the commutation relationship near the light cone, because in ordinary theory we already know that we can derive the whole theory if we know only the commutator in the immediate surroundings of the point $x=x'$, $t=t'$. So also here we can be quite satisfied with knowing the commutation relationship very near to this point, because all the rest can be derived by integrating the wave equation. That is, we can then proceed from the time t to time $t+dt$, and so forth, and thereby we can get the whole solution. We are interested only in the behavior near the light cone, and this behavior we can get from solving Eq. 6.

The Solutions

Now I do not want to go into the mathematics of the solution, but I would like to write the solutions in the form of a few pictures. If we solve the same problem for the linear wave equation, then, of course, we would also find for the function c just a linear wave equation. The term with the third power of c would be left out, and we would get as a solution for the anticommutator the well-known propagation function of Schwinger. Let us for a moment assume that we are not dealing with spinor particles but are dealing with scalar particles, and then we do not have to deal with the S-function of Schwinger but with the Δ-function of Schwinger, which is a function of the distance between x and x' only. This makes it easier to draw pictures, because then we have one single function of only one variable s as in the function shown in Fig. 2.

Vertically we have the Δ-function of Schwinger, and horizontally we plot the space-time distances between the two points. In the case of the linear theory this commutation function of Schwinger, the so-called Schwinger Δ-function, has the following property. It is a Bessel function for all finite distances s, and it is a Dirac δ-function just at the point $s=0$. So at the Δ-axis we have drawn the δ-function as s that would go to infinity, and the oscillating function for positive s is the Bessel function. (Fig. 2.)

This would be the solution of our wave equation for c, if the nonlinear terms were not present. Now we have to study the behavior in the case of the nonlinear theory, where we will first draw a picture for those cases in which the constant a_ν is still finite and not infinitely small. Then, of course, what we get is not the commutator but actually a sum of terms, the first of which is the commutator, and then there are higher terms, which, of course, somewhat change the picture, so that only in the limit for $a_\nu \to 0$ it will become the commutator. The picture will then look like Fig. 3.

For large space-time distances the function again will be nearly a Bessel function, because then the nonlinear terms are very small and do not have strong influence. But for small space-time distances the influence of the nonlinear terms is felt, and then there are some deviations from the old picture. Hence, out at the right we have the Bessel function again, but in the inner part it turns out that there are very fast oscillations so that the function starts oscillating quite rapidly near $s=0$, resulting in an infinitely frequent oscillation with infinite amplitude very near the origin. It is readily apparent that such a function can easily be integrated over the whole distance from $s=0$ to any finite value of s. If we now go to the case where the a_ν is exceedingly small, then this region of very fast oscillation moves always closer and closer to the origin, so finally we are left with a Bessel function for all finite values of s, and only in the origin do we have the fast oscillations (Fig. 4). This means that

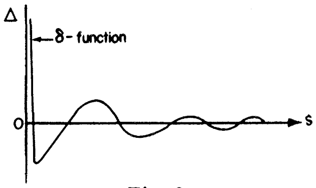

Fig. 2.

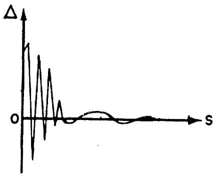

Fig. 3.

now our commutator in the case of the nonlinear theory does almost look like the commutator in the linear theory. The only difference is that the delta function at the origin disappears, and instead of the delta function we have an infinitely fast oscillation; that is, we have an essential singularity, and therefore the value of the function is not defined at this point, but the integral is defined and is always 0 if we only go close enough to the origin. Really *the only difference is the disappearance of the delta function.*

Now this actually helps a lot for the whole divergence problem, because as soon as one starts with the commutation function that has no delta function at the origin, all the divergencies disappear, and we do get a convergent theory.

Before going on, I must write one function that can be derived from the Schwinger propagation function S, which Schwinger calls the S_1-function. This S_1-function is derived from the propagation function S in the following way. One writes a Fourier expansion of the S_1-function, changes the sign of all the Fourier components of which the frequency has a negative sign, and gets a new function, which can be derived from the old one by an integral operation. This function, multiplied by i, is called the S_1-function, and theoreticians know the properties of this function. Here I have a special reason for writing it, and I will just do so in spite of the fact that at this moment it cannot be understood very well why it is useful:

$$S_1(x-x') = i\left[S^{(+)}(x,x') - S^{(-)}(x,x')\right] = \frac{\kappa^2}{4\pi}\gamma_\mu \frac{\partial}{\partial x_\mu} \mathrm{Im}\left[\frac{H_1^{(1)}(u)}{u} + \frac{2i}{\pi u^2} - \frac{i}{\pi}\ln\left|\frac{\gamma u}{2}\right|\right] - \frac{\kappa^3}{4\pi}\mathrm{Im}\left[\frac{H_1^{(1)}(u)}{u} + \frac{2i}{\pi u^2}\right]$$
$$\text{with } u = s^{1/2}\kappa;\quad s \lessgtr 0 \qquad (7)$$

The point is that this function of u or s has the one property that is important: namely, it contains terms that fall off slowly with distance. The terms in Eq. 7 not containing Hankel functions do *not* appear in Schwinger's work and are added here because of the omission of the Dirac δ-function from our S-function.

The next problem is: If we introduce such commutation relationships, have we any hope that this can lead to a consistent mathematical scheme of quantization? In order to explain why I believe that this can lead to such a scheme, I must go back to some of the mathematical fundaments of present-day quantum theory. I think one can understand these fundaments even if one does not go into the details.

In ordinary quantum theory, the commutation relationships would not be the ones that I put down here. As a matter of fact, in ordinary quantization of wave fields, one does start with rather similar commutation relationships; however, one includes the delta function at the origin, and thereby one gets into all the divergency difficulties. If one omits the delta function, as we are inclined to do here, one ruins the theory completely. That is, one makes in this way a complete change, and the problem is: What price has to be paid for it? Certainly one does not get such a change for nothing. There must be some very serious deviation from ordinary quantum theory. And this I can explain in the following way.

The fundamental difference between quantum theory and classical theory is that in quantum theory not only the actual state of a system is important but at the same time all possible states of a system. For instance, when one is calculating the normal state of a hydrogen atom, it is not sufficient to know that the electron moves in an orbit of radius 10^{-8} centimeter, but it makes a difference whether the hydrogen atom is in a very small volume or in a very big volume. The eigen-states really are different, depending on whether the box in which the hydrogen atom is contained is small or big. That is, the possibility of the atom to get to very great distances is involved in the calculation of the eigenvalue. Or in calculating the scattering of a particle, we usually calculate it with the help of so-called "intermediate virtual" states. These states actually never are occupied, and yet for the scattering it is a problem to know what are the intermediate virtual—that is, possible—states. Therefore, contrary to classical theory, all possible values for a certain quantity are important in the mathematical formulas.

Coming back to the quantization of waves, we say that not only such wave functions as actually do occur in nature are important for quantization of waves but also all "possible" wave functions. From this aspect one comes very easily to an almost absurd conclusion. If, one says, space and time are really continuous in a mathematical sense, then the wave functions of the following type also belong to the possible wave functions. Assume a wave function that has the value 1 at every point where the coordinates have rational values and has the value 0 at every other point. Such a wave function is pure nonsense from the point of view of the physicist. Still it would be difficult in normal quantum theory to exclude any possible wave function from the mathematical scheme, even if it has some absurd properties, for instance, infinitely fast oscillations. This situation is probably the root of the so-called "divergency difficulties."

How do these divergency difficulties occur in the ordinary mathematical scheme? We usually say that all the stationary states of a quantum theoretical system define a certain Hilbert space. For instance the states of the hydrogen atom can be defined as the vectors in the Hilbert space, and we use it in quantum theory. Here I think it is reasonable to divide the Hilbert space into two parts. We can say that all existing stationary states up to a certain maximum energy, or rather mass, of the total system may be called Hilbert space No. 1. All other states may be called Hilbert space No. 2. The limiting mass may be extremely big. Let us assume the whole mass of the universe. Then it is obvious that only rather smooth functions can be expanded by using the states of Hilbert space No. 1 only, and for the infinitely many other wave functions one would either need Hilbert space No. 2 for expansion or one could not represent them at all. On the other hand, the states of the Hilbert space No. 2 do not occur in nature, and therefore it may be possible to change the rules of quantum theory with respect to these states of the second kind. That is what I do when I omit the δ-function. This change is actually necessary for the following reason: One can calculate the commutator by first going from the vacuum to the first group of excited states and then back to vacuum. Then I go from vacuum to the second group of excited states and back to vacuum, and so on. In any of these cases from every transition I get a function of this type:

$$\langle\Omega|\psi_a(x)|\Phi\rangle\langle\Phi|\psi_\nu^+(x')|\Omega\rangle + \langle\Omega|\psi_\nu^+(x')|\Phi\rangle\langle\Phi|\psi_a(x)|\Omega\rangle$$

where Ω is the vacuum and Φ the intermediate state. This is to be summed over Φ for getting the vacuum value of the anticommutator of Eq. 2. Now it has been shown in papers by Gell-Mann and by Low and by Källen and by Lehmann that each group contributes a δ-function at the origin and that all these δ-functions at the origin do not cancel, but they add up. If one says that the δ-functions at the origin cancel, it means that one has given up quantum theory for Hilbert space No. 2. One has sacrificed this

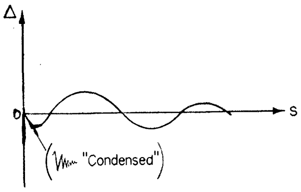

Fig. 4.

and, instead, has got some new mathematical scheme in which one has replaced the total Hilbert space by a thing which one may call a Hilbert space with a roof on top of it. I do not know whether this picture helps the mathematicians, but it shows the purpose of Hilbert space No. 2.

Having introduced this kind of Hilbert space, I am now far away from ordinary quantum theory but perhaps not too far from the experimental situation. So we replace Hilbert space No. 2 by a kind of imaginary Hilbert space, or by what I have called the roof on top of the Hilbert space.

The rest is just straightforward mathematics. So far we have had physical assumptions formulated in a mathematical language—assumptions about the physics of the problem—but from here on we have only mathematics. I shall not go into any calculations but shall just speak about the method and then give the results.

The method that can be used most conveniently is the so-called "new" Tamm-Dancoff method. It is a method that has been developed by Schwinger, Gell-Mann and Low, Freese and Zimmermann of Göttingen, and Goldberger, following an old paper of Tamm and Dancoff in 1941. So it is a rather well-known mathematical frame nowadays, and the great advantage of this frame is that one can work out a mathematical scheme in which one is interested only in matrix elements for those operators or products of operators that lead from the vacuum to a state of a finite energy. That is, one has to do only with matrix elements in Hilbert space No. 1, in which ordinary quantum theory shall be true. The whole contribution from the states of Hilbert space No. 2 comes only in the form of the commutation relationship. So it is quite sufficient for the calculation to know the behavior of the commutator near the point $x = x'$. This we have defined by means of the function given in Eq. 7. So we have actually a mathematical scheme by which we can calculate the energy eigenvalues, and it turns out that no divergency difficulties occur.

Now I will tell the results. One can ask: Are there stationary states; and, are there stationary states, say, with the spin ½, so that the angular momentum is $(\frac{1}{2})\hbar$? The result is that there is a lowest stationary state with spin ½, and the eigenvalue is given by putting κ (the energy or the rest mass of the system) equal to $7.45/l$. That it must have the factor $1/l$ in it is obvious, because l is a constant of the dimension of a length, and if $\hbar$ and c are made equal to 1, which is always done in these calculations, $1/l$ is the same as the dimension of a mass, and therefore this $7.45/l$ is just the eigenvalue of a mass.

We see that this equation leads to one particle, a fermion, which has this mass. If we assume that l is of the order of the Compton wave length of a π meson, which is a sensible assumption for this kind of a theory, then the mass of this particle turns out to be roughly that of the proton. Then one can also ask whether there are particles of Bose statistics, and of integer spin number. One can write the conditions for it, and actually one does get a kind of Bethe-Salpeter equation which leads to the existence of Bose particles. The mass for these Bose particles will again contain the factor $1/l$; and then it will depend on the numerical coefficient of $1/l$ for this Bose particle, whether it is stable or unstable. If the factor would turn out to be of the order of 20, then, of course, it would be unstable, because it could disintegrate into two of these fermions. If, however, the mass turns out to be only, say $1/l$, then it would be a stable particle and could correspond to the π mesons. (*Note added in proof*: Later calculations by Kortel, Mitter, and me have led to the eigenvalues $0.95/l$; $3.32/l$; $0.33/l$ and $1.74/l$ for the masses of Bose particles.) Strangely it turns out in the calculations, as far as we could see (but the calculations are not quite finished yet), that apparently one solution for the rest mass of the Bose particle is just 0. That is, one gets something like light quanta out of this calculation. The reason can be seen as follows. One can start with a different question and then it is easier to judge the answer. What is the interaction between two of the fermions we found? That, of course, follows again from the wave equation. All the interactions are defined by this wave equation. So one has just to calculate what happens when two such fermions are scattered by each other. And it is calculated by normal application of the Tamm-Dancoff method. Then it turns out that between two Fermi particles we have a long-range force of the Coulomb type —a force where the potential energy drops off as $1/r$. This long-range force is connected with the term

$$\frac{\kappa^2}{4\pi}\,\gamma_\mu\,\frac{\partial}{\partial x_\mu}\,\mathrm{Im}\left(-\frac{i}{\pi}\ln\left|\frac{\gamma u}{2}\right|\right)$$
$$\text{wit } u = s^{1/2}\kappa$$

appearing in our new S_1-function as given by Eq. 7. That is, just the very fact that one has omitted the δ-function in the commutator produces in the S_1-function an additive term that only very slowly decreases as the distance increases. For as I said before, these terms in Eq. 7 added to the Hankel function are the direct consequence of the omission of the δ-function at the origin. The scattering can be calculated in a very rough approximation with the Feynman graph shown in Fig. 5. Say we have two such fermions coming in, then we have to assume interaction through two more such fermions, and finally two come out. If one calculates this Feynman graph, then between these vertex points one has to put in the S_1-function, and therefore one gets an interaction of the Coulomb type. Really this Feynman graph is not a good approximation, so the calculation has to be done more carefully.

So, it seems that this equation has Bose particles of the rest mass 0 as eigenvalues, and this is, of course, a very interesting contribution to the problem of the elementary particles, because it shows that also the light quanta in the real theory of elementary particles may have to do with just this singularity of the S-functions at the origin. If the model of the elementary particles that is formed by the theory is correct, it would mean that the Coulomb forces—the electromagnetic forces—are for nature that method by which nature avoids divergency difficulties, which otherwise are always met in the theory.

Next Steps

The next problem is to calculate the higher boson states and also to calculate for them the corresponding value of $g^2/\hbar c$ and to find whether these quanta are scalar or vector quanta, and so forth. All this is just now in progress, so I cannot report the result. I just want to mention a few problems that are important and should be solved before one can take such a model quite seriously.

One of the most important questions will be: Does there exist some invariance property that corresponds to the gage invariance in electrodynamics? Only if this gage invariance is actually present, does one have a real analogy to the experimental situation. This gage invariance is, of course, decisive for the conservation of charge, the determination of $\frac{e^2}{\hbar c}$, and so forth. It may be that the gage invariance comes out by itself in the theory. This would be extremely interesting, if it were so. It may also be that it restricts the possible assumptions about the main interaction term. This would also be a very interesting result.

The next step would be to calculate the masses of the different Fermi particles and the different Bose particles and see whether that has any resemblance to the actual elementary particles.

This is the general picture of what I wanted to tell. I would like to add a few remarks about the difference between such a scheme and what one has hitherto done in the theory of elementary particles, especially in the quantization of wave fields. Usually in the quantization of wave fields one says: We have free particles, and there is an interaction. We first assume that the interaction is small, and then later we try also to account for strong interactions. Here we see that such an assumption would be complete nonsense. There is absolutely no meaning in saying: Let us first assume that the coupling, the nonlinear term, is small. If we would assume that this constant l, which has the dimension of a length, is small, this would not change anything in the theory at all, because it would just make a similarity transformation in the whole theory. That is, all masses would become bigger proportional to $1/l$, but the spectrum of these masses would not be changed. So it just means that the dimension of the whole world would change, but the eigenvalues and the whole spectrum—all that—would not be changed. Therefore in such a theory the idea of small interaction is just nonsense. Also the idea of free particles that have no interaction in a first approximation is nonsense in such a theory, because the particles are found in exactly the same mathematical frame in which all the interactions are found. That is, in such a theory, not only all the masses of the particles would be determined, but at the same time all the interactions would be determined. Therefore, in such a theory it is quite obvious that one would get a definite value, for instance, for the finestructure constant $e^2/\hbar c$. Then, one may say: Is there not a danger that one still will come to some contradiction in such a theory, for instance, with respect to the law of causality? May it not be that, on account of introducing this rather strange commutation function, one gets deviations from causality, which then lead to a scheme that we cannot accept from the experiments? On the other hand, we have introduced the commutator from the closest possible analogy with the propagation functions, and the propagation functions, since they are calculated classically, are functions that obey the ordinary rules of causality. Therefore I put it that it seems as least unlikely that one gets into trouble with deviation from the causality, although one must admit that the theory is not yet so well studied that one can be absolutely certain.

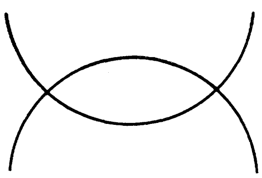

Fig. 5.

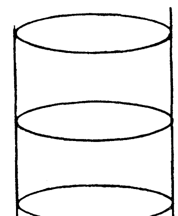

Fig. 6.

There may also be some difficulties with the convergence of the mathematical scheme, but in any case one can say that all those divergency difficulties, which one knows normally from the quantization of waves, do not occur here. Whether other difficulties may occur—say, the question whether the Tamm-Dancoff method converges—is a different matter and remains to be seen.

So, generally speaking, I certainly do not say that this is already a good and sensible model for the theory of the elementary particles, but I would like to say that, whatever one will in the future do for obtaining a theory of elementary particles, one will have to look in a similar direction as here. That is, one will have to look for a theory in which one does not start from the wave equation for the mesons or the nucleons or anything like that, but one will have to start with an equation for matter only, and one will have to try to derive all the different masses of the elementary particles from just one wave equation, of which these masses shall be the eigenvalues. So I think this tendency toward such a theory is almost necessary, but as to the exact form in which one will gradually be induced to give it, one will have to be led by what will come out of it.

Discussion. In what sense can one say that "fermion states" or "bosom states" are found from Eq. 1? Answer: Such states are obtained from the vacuum state Ω by operating on it by an odd or an even number of factors ψ or ψ^+. Maybe a meson interaction between fermions will look, as a Feynman diagram, like a ladder with loops. (Fig. 6). Again let me stress that the limit $l \rightarrow 0$ is meaningless; the theory goes over into "ordinary" quantum theory in problems of low energy where one does not consider transitions to virtual states of high energy, because one stays inside Hilbert space No. 1 anyhow.