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Mach Rings Verified by Numerical Differentiation

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The Mach ring effect is well known to certain psychologists and physicists and has been recorded as a part of the technical literature since 1865 (*1*). It is probably not so well known to others working in related fields. Mach established very definitively, using black-and-white patterned sector disks and rotating drums, as well as photographs of black-and-white line drawings, that the brightness of a perceived area is not simply a direct effect of the luminous intensity of local stimulation, but that it also depends on the distribution of stimulus luminances over a much larger area. Mach himself stated, "It is easy to go wrong in judging of the objective distribution of light according to the subjective impression." To illustrate the ubiquity of the effect, he summarized, "We came across the same phenomenon in the investigation of shadows, and of spectral absorption, and in countless other cases" (*2*, p. 219). The intent in this article is to report what is apparently an unusual source of the Mach ring effect and to suggest a well-known mathematical technique that can be applied generally to verify the effect with no limitation on the form of the stimulus luminance distribution.

Mach Rings in Electron Diffraction Patterns

This unusual source of the Mach ring effect is a modern technological one in which microdensitometric traces are made of the electron diffraction patterns of certain gas molecules. Typically, a film is exposed to the diffraction pattern, then processed. When the processed film is illuminated diffusely from behind, a characteristic series of concentric rings of varying widths and brightnesses may be seen; these are used to identify the particular gas molecule. The film is then placed in a microdensitometer and the varying luminous density across a diameter of the circular pattern is measured. In the present case, light and dark rings were seen on the illuminated film that did not bear a direct relationship to the luminance variations recorded on the densitometric trace.

The undulating curve in Fig. 1 is a smoothed drawing of the logarithm of the luminance (negative density) variations across a radius of the circular film pattern. The flat portion of the curve at the extreme left represents the limit of sensitivity in the microdensitometer. Higher portions of the curve represent areas of greater light transmittance through the film, and lower portions represent areas of lesser transmittance. The square-cornered curve superimposed on the luminance trace in Fig. 1 is a radial cross section representing the measured location and a roughly quantitative judgment of the relative brightness of the rings visible through the illuminated film. To correspond with the visual implications of the luminance curve, higher portions of this curve represent lighter rings and lower portions represent darker rings. If brightness bore a direct relationship to measured luminance, then large dips and rises like those in the brightness curve should have been found at corresponding positions in the log luminance curve. The fact that these large dips and rises did not appear was, indeed, puzzling until we were consulted for a possible explanation of the rings. A careful glance at the shape of the log luminance curve at positions (indicated by arrows) corresponding to the dips and rises in the brightness curve showed irregular acceleration in curvature. This immediately suggested the stimulus situation for the Mach ring effect.

The Stimulus for Mach Rings

Koffka (*3*, pp. 169–171) has described this situation in terms of the curves shown in Fig. 2. Where luminance varies uniformly from one side of an area to the other, as in curve *a* of Fig. 2, the stimulation at point *p* is the same as the average stimulation in its neighborhood, and a uniform brightness may be observed over the entire area. (The appearance of uniform brightness across an area having uniform variation in luminance is a well-established fact, not generally known, that dates back at least to Mach's report in 1865.) When, however, the luminance variation across the area is not uniform, as in curves *b* and *c* of Fig. 2, something else happens. The point *p* has a higher luminance in curve *b*, and a lower luminance in curve *c*, than the average of the points on both sides of *p* in the near vicinity. Under these conditions, and when the difference between the *p* luminance and that of the average of the neighboring points is large enough, the Mach effect appears. When the *p* luminance is greater than the average luminance of nearby points, a line will appear at *p* that is lighter than the neighboring areas; when the *p* luminance is less than the neighborhood average, a line will appear at *p* that is darker than the surrounding area. Typical Mach rings appear instead of lines when these luminance differences are found in concentric circular patterns.

Mach (*1*) has shown mathematically that the location and relative brightness of the rings can be predicted from the sign and magnitude of the second derivative of the luminance curve plotted as a function of distance across the area concerned. When the sign of the second derivative is positive, a dark ring appears, and when it is negative a light ring appears so long as the value of the second derivative is above some critical value.

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Ludvigh (*4, 5*) has recently refined this analysis to account for the edges, widths, and separation of the rings in terms of the second and fourth derivatives of the luminance function.

Application of Numerical Differentiation

In Mach's earlier studies, and in more recent studies by Thouless (*6*), Koffka and Harrower (*7*) and Ludvigh (*4, 5*) energy distributions were used for which it was necessary to find substitute expressions for the second derivative because of infinite values at points of discontinuity. In the present case, although the luminance curve is continuous, the equation for the curve is not known. The second derivative may, however, be obtained directly from a tabulation of the ordinates of the luminance curve by the method of numerical differentiation. With certain simplifications, this solution can be easily worked out with a desk calculator. The solution consists first of choosing an appropriate interpolation formula, next taking derivatives of the formula, and finally applying the resultant formula for the second derivative to the function under consideration. Stirling's formula, (*8*, p. 70) was used for this purpose, but the results were later checked by Bessel's formula (*8*, p. 74) and found to be comparable. The second derivative of Stirling's interpolation formula for any value of the abscissa x is

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \left(\frac{12u^2-2}{4!}\right)\Delta^4 y_{-2} + \left(\frac{20u^3-30u}{5!}\right)\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right) + \left(\frac{30u^4-60u^2+8}{6!}\right)\Delta^6 y_{-3} + \ldots\right]$$

Where

$$u = \frac{x - x_0}{h}$$

x_0 are the points on the abscissa where readings are taken, and h is the distance between readings (*8*, p. 129). If second derivatives of this function are required only at x_0, then $u = 0$, which simplifies the equation considerably. The equation may then be converted back into the original ordinates so that differences need not be used at all. A second simplification can result from converting the physical units of the abscissa into arbitrary units such that $h = 1$. The equation (using all terms through and including the sixth difference) then becomes

$$\frac{d^2y}{dx^2} = 0.011y_{-3} - 0.150y_{-2} + 1.501y_{-1} - 2.724y_0 + 1.501y_1 - 0.150y_2 + 0.011y_3.$$

This reduces the computing problem to one that can be readily solved on a desk computer. Although the results represent an approximation, they were adequate to verify the basic hypothesis; the technique may be useful to others who wish to extend the investigation of Mach rings to areas having irregular luminance variations.

Mach Rings and the Second Derivative

Results of the numerical differentiation are shown in Fig. 1, where the second derivative of log luminance has been obtained every 4 millimeters along the abscissa of the log luminance function, starting at the point that represents the limit of sensitivity of the microdensitometer. According to Ludvigh (*4*), the two edges of a ring appear symmetrically on each side of a second derivative maximum of appropriate magnitude. Ring locations have been designated by arrows in Fig. 1. Positive maxima may be predicted at the particular positions around which there is positive acceleration (as in curve *c* of Fig. 2) and negative maxima may be predicted at those positions around which there is negative acceleration (as in curve *b* of Fig. 2). Arrows pointing up indicate positions at which negative maxima would be predicted in the second derivative (lighter rings) and those pointing downward indicate positions at which positive maxima (darker rings) would be predicted. A comparison of the regions where the second derivative is most deviant from zero with the deviant regions of the related brightness curve gives a reasonable indication of the fact that the ring centroids are associated with the highest values of the second derivative.

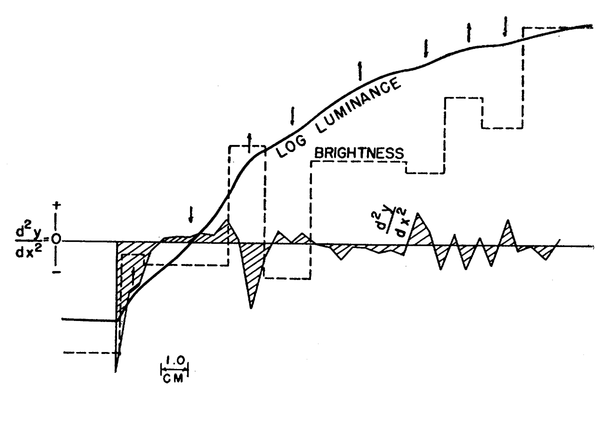

Fig. 1. Log luminance curve of radial cross section of electron diffraction pattern, brightness curve through same cross section showing relative brightness of Mach rings, and second derivative function of log luminance curve derived by numerical differentiation.

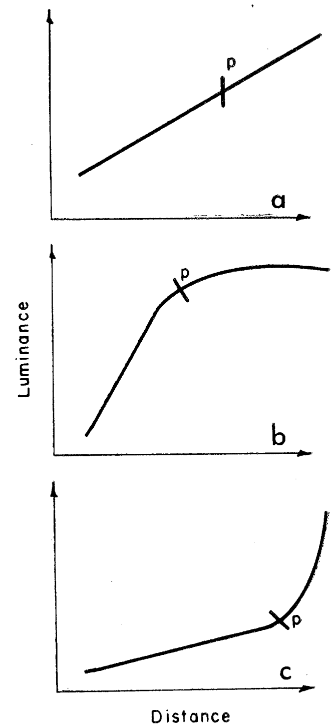

Fig. 2. Curve *a*, constant luminance gradient; curve *b*, negatively accelerated luminance gradient; curve *c*, positively accelerated luminance gradient.

What is not so clear by this approximation technique is the actual location of some of these rings. In tabulating data from a curve such as this, it is difficult to obtain data that are accurate to more than three significant digits. In so doing, small discontinuities are artificially introduced that show up as marked increases or decreases in the resultant derivatives. This is particularly true in the regions of the curve that show small changes in the first place. The oscillation of the second derivative represents, to some extent, the inherent variability of the method and can conceivably mask some of the differences being studied. To minimize this interference as much as possible, care should be taken in the tabulation of the data. The ordinates in this case were obtained by greatly expanding the scale on which the original data were plotted and then smoothing the resultant curve; the result was that the ordinates could be read off to three-digit accuracy. Additional distortion can result from too many or too few measurements per unit length of the abscissa.

Although these limitations exist in numerical differentiation, it does not appear to be a great handicap in this example. Some of the rings are not clearly identified by sharp deviations in the second derivative, but usually the second derivative had a constant sign in the region of these rings except at the right end of the second derivative curve. Even there the preponderance of points was in the proper direction. It is conceivable that a closer association of rings with second derivative maxima could be found if the log luminance function had been transformed to visual units (such as Munsell value) before differentiating, since equal visual brightness units are not a direct logarithmic function of luminance over the entire range of visual brightnesses; a quintic parabola gives a closer fit. For our purposes, however, the association was close enough to verify the Mach hypothesis.

The surprising thing in this instance was the clear appearance of rings for what seem to be only minor undulations in the log luminance curve; reports in the literature have been concerned usually with somewhat larger gradient changes.

A possible, more general application of numerical differentiation to problems of this type is suggested because continuous second or higher derivative functions may be obtained from any irregular luminance distribution. Any conclusion regarding the complete validity of this application must depend, however, on a more systematic appraisal using refinements such as those recommended by Ludvigh (*4*) for computing the effective distribution of the proximal visual stimulus.

References and Notes

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Alfred F. Huettner, Scientist and Teacher

Alfred Francis Huettner, professor emeritus of biology at Queens College, died 27 September 1955 in his 73rd year at his home in Douglaston, Long Island, New York. He was born at Reichenbach, Germany, on 13 December 1882 and came to the United States in 1904. Neither in his youth nor later was he afraid of hard work in any form. He was largely self-supporting and worked at a variety of jobs, including working as a section hand and shipping on as a sailor. It may well be that his early hardships fostered the keen sympathy and interest he showed later for his students, many of whom had to work for their own support.

He was delayed in his college work and graduated with a B.A. degree in 1916 from the University of South Dakota. In the fall of 1916, he entered Columbia University to do his graduate work in zoology under E. B. Wilson; the degree of doctor of philosophy was conferred on him in 1923. His dissertation, "The origin of the germ cells in *Drosophila melanogaster*," was a significant cytological work of technical difficulty and proved the multiple origin of the germinal nuclei in the fruit fly. This explained unexpected genetic findings in somatic mutation. Previously it had been supposed that the isolation of the germ cells in *Drosophila* was similar to that found in earlier work on *Miastor* and on *Chironomus*.

Dr. Huettner continued teaching and research on the staff of Columbia University as instructor and assistant professor until 1932, when he joined the biology department at Washington Square College of New York University as associate professor. He was promoted to professor in 1936. He joined the biology department of Queens College, Flushing, New York, in 1938, the second year of the college. There he organized the instruction in vertebrate zoology and vertebrate embryology. He retired in 1952 after serving for 7 years as chairman of the department. During all his years of active duty at Queens College, he also carried the burden of the premedical and predental committee chairmanship. He did not spare himself in this work, nor in any other, and rendered great service to the college and its students. It was quite fitting that on his retirement the Board of Higher Education honored him by naming him the first professor emeritus of Queens College.

Following his first cytological work already referred to, Dr. Huettner continued with cytological studies on the chromosomes and on the central bodies of the fruit fly. Possibly of more significance was his work on the early embryological stages of *Drosophila* that was carried out with the active collaboration of students. Benjamin Sonnenblick completed one phase of this project, which was published as a chapter in *Biology of Drosophila*. Dr. Huettner's last major contribution to science and teaching was achieved with the publication of his book, *Fundamentals of Comparative Embryology of the Vertebrates*. This book is still very widely used and is unique in the excellence of its three-dimensional illustrations, most of which were his own drawings based on his own preparations. Much original research went into the preparation of this book.

The admirable reputation of Dr. Huettner as a teacher was based on his wide knowledge of the biological sciences and his enthusiasm for the subject, which he was unusually successful in imparting to his students. His personal interest in students won him their esteem and affection. He interrupted his labors to give his time and help to colleagues and friends as well as to students. He was a man who set high standards of moral and intellectual integrity for himself and was able to live up to them.

Dr. Huettner enjoyed working at his home and in his garden; he became proficient in the laying of concrete walks and walls. He enjoyed the distinction of having grafted seven kinds of fruit on one tree and lilacs on his privet hedge. He was a man of many talents.

Dr. Huettner was a member of Sigma Xi and several scientific societies.

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