Fifty Years of Boundary-Layer Theory and Experiment

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THE year 1954 marked the 50th anniversary of the Prandtl boundary-layer theory from which we may date the beginning of man's understanding of the dynamics of real fluids. A backward look at this aspect of the history of the last 50 years may be instructive. This paper (*1*) attempts to compress the events of those 50 years into a few thousand words, to tell in this brief space the interesting story of the development of a new concept, its slow acceptance and growth, its spread from group to group within its country of origin, and its diffusion to other countries of the world.

The original brief paper of Prandtl (*2*) was presented at the Third International Mathematical Congress at Heidelberg in 1904 and published in the following year. It was an attempt to explain the d'Alembert paradox, namely, that the neglect of the small friction of air in the theory resulted in the prediction of zero resistance to motion. Prandtl set himself the task of computing the motion of a fluid of small friction, so small that its effect could be neglected everywhere except where large velocity differences were present or a cumulative effect of friction occurred. This led to the concept of boundary layer, or transition layer, near the wall of a body immersed in a fluid stream in which the velocity rises from zero to the free-stream value. It is interesting that Prandtl used the term *Grenzschicht* (*boundary layer*) only once and the term *Übergangsschicht* (*transition layer*) seven times in the brief article. Later writers also used *Reibungsschicht* (*friction layer*), but most writers today use *Grenzschicht* (*boundary layer*).

Prandtl made a few qualitative experiments in a water tank 1.5 m long with a partition midway between the surface and the bottom and a paddle wheel at one end to produce flow. Photographs were taken of mica particles on the surface of the water illustrating separation around various obstacles. In one experiment he removed the boundary layer by suction through a slot, demonstrating that separation occurred only when a boundary layer was present.

The first decade produced only seven papers, all from the Göttingen group. Three graduate students made computations of the flow: Blasius (*3*) for a flat plate and Boltze (*4*) for a body of revolution in 1908, and Hiemenz (*5*) for separation on a circular cylinder in 1911. Prandtl (*6*) applied the concept to heat transfer in 1910. Töpfer (*7*), a mathematician, refined the computations of Blasius. In 1914 Prandtl (*8*) explained the critical Reynolds number of the sphere, which had been observed by Eiffel, as due to transition of the flow in the boundary layer from laminar to turbulent.

Thus, in the first 10 years, there were five workers in the boundary-layer field, all at Göttingen. Th. von Kármán, who came to Göttingen in 1906 and was graduate assistant from 1908 to 1912, gives us a delightful anecdote of those days in his most recent book, *Aerodynamics—Selected Topics in the Light of Their Historical Development* (Cornell Univ. Press, 1954). He tells how Hiemenz found oscillating flow behind the cylinder on which he was trying to measure pressure distribution and separation point for comparison with boundary-layer theory. Von Kármán thought that if the flow always oscillates, the phenomenon must have a natural and intrinsic reason. He thus became interested in the stability of the vortex street. In all this period there were no direct observations or experiments on the boundary layer itself. Every result was indirect, inferred from the gross aspects of the flow.

World War I produced a blank in the record of publications on boundary layers. G. I. Taylor, quite independently of Prandtl's boundary-layer theory, in a paper (*9*) written in 1916 but given general publication in 1920, applied the concept of the laminar sublayer in the turbulent flow in a pipe or near a plate to problems of heat transfer and skin friction. Prandtl lost his graduate students to military service. Von Kármán went to Aachen in 1912 but returned to Austria for the period of the war. Following the war, the Aachen group under von Kármán took up boundary-layer theory. The *Zeitschrift für Angewandte Mathematik und Mechanik* was founded in 1921, and its first volume contains five papers on boundary layers. Many of them were the result of work completed during the war. Von Kármán (*10*) proposed his well-known integral equation for approximate computation of the development of boundary layers along a surface, and K. Pohlhausen (*11*) applied the method to several cases, using a polynomial approximation for the velocity distribution. E. Pohlhausen (*12*) computed heat transfer. Prandtl and his group became interested in the origin of turbulence, beginning a program that was to continue intensively for many years. Noether (*13*) summarized the state of knowledge on this subject at the time. Tietjens' dissertation (*14*) on stability of laminar flow was written in 1922, and Prandtl (*15*) published a paper on the subject. In this year the precursor of the Congresses of Applied Mechanics was held in Innsbruck. The first formal congress was held at Delft in 1924. Here Burgers (*16*) reported the first experimental observations of the velocity distribution within the boundary layer of a plate. They were made by his student, B. G. van der Hegge Zijnen (*17*). Von Kármán (*18*) gave a general lecture on the origin of turbulence. Heisenberg's paper (*19*) on stability of flow appeared. Tollmien's dissertation (*20*) was published. It dealt with the growth of the laminar boundary layer on a rotating cylinder on suddenly starting from rest.

Thus in the second decade, six years were lost as a result of the war. But now three groups, Göttingen, Aachen, and Delft, were interested. The origin of turbulence was the key theoretical problem. *Zeitschrift für angewandte Mathematik und Mechanik* was started. The first experiments had been made on the boundary layer itself. It had required 20 years to reach this point.

In 1925 Dönch's experiments (*21*) on divergent and convergent channels appeared, Tietjens' work (*22*) was published, and Bairstow (*23*) recomputed the boundary-layer flow of a flat plate. In 1926 Tollmien (*24*) tackled the problem of the turbulent wake, and Ackeret (*25*) that of control of the boundary layer by suction. The Second International Congress for Applied Mechanics was held at Zurich, and von Kármán visited the United States. In 1927 von Mises (*26*) suggested the use of the stream function as one of the independent variables in boundary-layer computations, and Prandtl (*27*) published various formulas for skin friction of a flat plate computed from data on pipes.

In 1928 Hansen (*28*) at Aachen repeated the Delft experimental measurements of velocity distribution in the laminar boundary layer of a plate. Thom (*29*) and Fage (*30*) in England computed the boundary layer over the front part of a cylinder at various Reynolds numbers. B. M. Jones (*31*) began his interest in boundary-layer problems in a classic paper on "The streamline aeroplane." Schrenk (*32*) made experiments on wings with boundary-layer suction. In 1929 Prandtl turned to the influence of stabilizing forces on turbulence (*33*) and the role of turbulence in technical hydrodynamics (*34*). Elias (*35*) made experiments on heat transfer on a plate. Levi-Civita (*36*) of Italy wrote on some consequences of the boundary-layer theory. Two British papers (*37*) showed the increasing interest in Great Britain. Tollmien's paper (*38*) on stability of laminar boundary layers appeared. Nikuradse's work (*39*) on convergent and divergent channels was published.

The year 1930 was the year of the Stockholm Congress. I am told by a friend who was at Aachen at the time that the Aachen group worked intensively in preparation for the congress, feeling a keen sense of competition with the Göttingen group and an expectancy of important results. Von Kármán's paper (*40*) on "Mechanical similarity and turbulence" appeared during the year, and a new computation (*41*) of turbulent skin friction on a plate was presented at the congress, under the same title. Prandtl had obtained essentially the same formula, but von Kármán's work was published first. In the fall of 1930 von Kármán became director of the Guggenheim Aeronautical Laboratory at the California Institute of Technology.

In 1930 there also appeared papers by Betz (*42*) and Wilcken (*43*) on the turbulent boundary layer on curved walls, Schlichting's work (*44*) on wakes, and measurement of heat transfer by Elias (*45*). Goldstein (*46*), and Falkner and Skan (*47*) in Great Britain presented the so-called "similar" solutions of the boundary-layer equations. Green (*48*) made further computations for the cylinder. The group at the National Bureau of Standards entered the field with a paper (*49*) on the effect of turbulence in wind-tunnel experiments, using the compensated hot-wire anemometer described (*50*) earlier.

In 1931 the first survey of the field was published by Tollmien in the *Handbuch der Experimental Physik*. Buri (*51*) and Gruschwitz (*52*) tackled the development of the turbulent boundary layer when it was subjected to pressure gradients. Falkner and Skan (*53*) and Fage and Falkner (*54*) presented improved methods for the laminar boundary layer under pressure gradients. Betz (*55*) commented on the von Kármán similarity hypothesis ,and Schrenk published another paper (*56*) on boundary-layer suction.

In 1932 Prandtl (*57*) further treated the turbulent boundary layer, Schlichting (*58*) wrote on the stability of the Couette flow, periodic flows in boundary layer (*59*), and the origin of turbulence in a rotating cylinder (*60*). E. Schmidt (*61*) published schlieren pictures of thermal boundary layers. G. I. Taylor (*62*) produced his vorticity and heat-transport theory. Clark Millikan (*63*) gave the theory of the boundary layer on a body of revolution. Crocco (*64*) applied boundary-layer theory to heat-transfer problems.

Schlichting's comprehensive paper (*65*) on the origin of turbulence appeared in 1933. This year also marked the demonstration by Schiller and Linke (*66*) of transition in the free vortex layer behind a cylinder at low Reynolds number. Schlichting (*67*) computed the spread of a laminar jet, and Goldstein (*68*) treated the problem of the two-dimensional wake.

In the last year of this decade, the Institute of the Aeronautical Sciences was founded, the Durand volumes on *Aerodynamic Theory* appeared, and the Fourth International Congress for Applied Mechanics was held at Cambridge, England. The first flight measurements on a boundary layer were reported by Stüper (*69*). Lyon (*70*) in England and Moore (*71*) in the United States described further work on the boundary layer on a body of revolution. Fediaevsky's (*72*) and Frankl's (*73*) papers on the turbulent boundary layer were published. Schmidbauer (*74*) computed turbulent boundary layers on curved walls. Von Kármán and Mullikan (*75*) published a theory of the laminar boundary layer with separation. I presented to the Congress of Applied Mechanics new experimental measurements (*76*) on boundary-layer flow near a plate, including hot-wire measurements of velocity fluctuations.

In this third decade there were five or six papers per year. Interest had spread to Great Britain, the United States, and Italy. Progress had been made on the analysis of laminar and turbulent boundary layers under pressure gradients, on analysis of the boundary layer on bodies of revolution, on jets and wakes, and on the theory of stability. Additional experimental work had been accomplished on the details of boundary-layer flow on flat plates.

The fourth decade began with the Volta Congress in 1935, at which the first paper on the boundary layer in compressible flow was presented by Busemann (*77*). During this year Tollmien (*78*) described the influence on stability of an inflection in the velocity-distribution curve, and Schlichting (*79*) computed the amplitude distribution in unstable boundary-layer oscillations. Tomotika (*80*) computed the laminar boundary layer on a sphere. Howarth (*81*) improved the calculations for the cylinder. A most important paper (*82*) was Schubauer's on measurements of air flow in a separating laminar boundary layer on an elliptic cylinder. These measurements became the crucial sieve to separate good from bad approximations in laminar boundary-layer computations. An ellipse of axis ratio 2.96 to 1 has become internationally known as Schubauer's ellipse. Almost immediately the theory of von Kármán and Millikan was applied to it by Millikan (*83*) and by von Doenhoff (*84*) independently.

In 1936 the Volta Congress report became available with von Kármán's ideas on the compressible boundary layer. Homann (*85*) solved the problem of flow at a stagnation point. Goldstein and Rosenhead (*86*), Preston and Piercy (*87*), and Howarth (*88*) continued the development of methods of computation for laminar boundary layers. The National Bureau of Standards published the full report (*89*) on experimental measurements of the flat-plate boundary layer. G. I. Taylor (*90*) gave a theory of the effect of turbulence on transition in boundary layers.

In 1937 there appeared an extensive report (*91*) from the National Bureau of Standards group on the effect of intensity and scale of turbulence which confirmed Taylor's theory and by inference contradicted the stability theory of the Göttingen group. The effect of curvature on transition was reported (*92*) from experiments by the Clauser twins at the California Institute of Technology. Frankl and Voishel (*93*) attacked the compressible turbulent boundary layer. Hartree (*94*) and Sutton (*95*) continued the perennial subject of laminar boundary-layer calculations.

In 1938 the Goldstein volumes, *Modern Developments in Fluid Dynamics,* came upon the scene as the compendium of existing knowledge. The Fifth International Congress of Applied Mechanics was held at Cambridge, Massachusetts. Melvill Jones (*96*) gave the first Wright Brothers lecture on "Flight experiments on the boundary layer." Von Kármán and Tsien (*97*) treated the boundary layer in a compressible fluid.

World War II apparently did not interrupt the development, although the dissemination of information was largely restricted to the country of origin. It becomes impractical to continue to list all of the important contributions year by year. In 1939 I gave the second Wright Brothers lecture (*98*) on "Turbulence and the boundary layer," a review of the contributions of the group at the National Bureau of Standards up to that time In 1940 there were many important developments in Germany not known in other countries until much later. Hohlstein and Bohlen (*99*) developed improved methods for laminar boundary-layer calculations. Hantzsche and Wendt (*100*) made further progress with compressible laminar layers, Schlichting (*101*) with stability theory for boundary layers with pressure gradient, Görtler (*102*) with stability of flow past curved walls. Schultz-Grunow (*103*) made experiments that demonstrated the lack of equivalence of turbulent flat-plate and pipe-velocity distributions by careful experiments for the two cases. Prondtl and von Kármán has assumed the distributions to be the same. In Switzerland Ackeret, Ras, and Pfenninger's work (*104*) on boundary-layer control through suction slots was disclosed.

In 1941 Emmons and Brainerd (*105*) applied high-speed computers to the compressible laminar boundary-layer problem. Stability-theory effort continued in Germany in 1941 and 1942 with applications to boundary layers with pressure gradient (*106*) and suction (*107*).

In 1943 Schubauer and Skramstad published in a confidential report (*108*) their classic experiments which fully confirmed the stability theories of the Göttingen school when the turbulence of the wind-tunnel air stream is low. This report was not made available in unclassified form until 1946. Mangler (*109*) discussed the "similar" solutions; Liepmann (*110*) described experiments on transition on curved walls.

In 1944 Iglisch (*111*) and Ulrich (*112*) computed the laminar boundary layer on a plate with suction. Schaefer (*113*) described solutions for a plate with suction with power law and exponential free-stream velocity variation with distance. Wieghardt (*114*) checked experimentally that increasing turbulence for flow near a plate would change the velocity profile of plate to that of pipe flow.

In the fourth decade there were about 14 papers per year. The laminar boundary-layer development and its stability were fairly well understood with correlated theoretical and experimental results. The turbulent boundary layer was not understood, but empirical procedures were available. Some empirical information on transition had been obtained. Numerous investigations had been made of the influence of compressibility. Boundary-layer suction for drag reduction was under active study. Only two-dimensional boundary layers and boundary layers on bodies of revolution had been treated.

The work that has been discussed so far is primarily that thought worthy of mention in Durand's *Aerodynamic Theory,* Goldstein's *Modern Developments in Fluid Mechanics,* and the recent book *Grenzschichttheorie* by Schlichting, published in 1951. To obtain a picture of the more recent effort, the critical reviews in *Applied Mechanics Reviews* were inspected. An attempt was made to select the work that dealt with theory and with experiments aimed at understanding the boundary layer itself, but admittedly the selection of titles is somewhat arbitrary. To these were added more recent papers in the current journals. The total was 561 papers, beginning with 43 in 1948, and 40, 68, 87, 95, 100, and 128 (part of 1954) in the succeeding years. This leaves a gap of a few years between the end of the first 40 years and the establishment of *Applied Mechanics Reviews* which has been covered by reference to Schlichting's book and other sources. The remaining discussion is by subject headings that are not mutually exclusive but are suggestive of the major fields of activity.

Strange as it may seem, the story of the laminar incompressible boundary layer is not exhausted with about 70 papers in this area reviewed in 6 years. Schlichting adopted Howarth's 1938 computation of the Blasius distribution for the flat plate without pressure gradient so that this bit may be regarded as closed. However, Mangler (*115*) proposed a two-parameter method for approximate calculations for the pressure-gradient case, and Wieghardt (*116*) carried through the procedure using both momentum and energy theorems and a polynomial of the 11th degree for the velocity distribution. Walz (*117*) proposed to return to a one-parameter family by abandonment of one of the boundary conditions that had always been regarded as essential. There has been little new experimental work, but the fine interferograms of Ladenberg and Bershader (*118*) should perhaps be mentioned.

The laminar compressible case continues to be of great interest. Again the papers deal mainly with approximate theories. Variable fluid properties have been introduced into the computations and further application made of high-speed computing machines. About 60 papers have been reviewed in *Applied Mechanics Reviews,* but of these only five present experimental data.

The turbulent incompressible boundary layer receives considerable attention with about 45 papers. Experimental investigations of the National Bureau of Standards group under Schubauer (*119, 120*), of Townsend (*121*) at Cambridge, England, and of Ludwieg and Tillmann (*122*) in Germany have contributed much to our understanding of the structure and statistical aspects of turbulent boundary layers. There have been some refinements in empirical methods of calculating their development under pressure gradients.

About 35 papers on the turbulent compressible boundary layer were reviewed. Mention may be made of the similarity theories of Lin and Shen (*123*) and of the study of integral methods by Libby, Morduchow, and Bloom (*124*). The theoretical and experimental work by Chapman, Rubesin, and their coworkers (*125*) has added much to our knowledge of this field.

The phenomena of separation in both laminar and turbulent and in both compressible and incompressible boundary layers, with and without heat transfer, although not in all combinations, account for some 30 papers. The experiments of Ackeret, Feldmann, and Rott (*126*) and of Liepmann (*127*) are now well known, as are the experiments of Schubauer and Klebanoff (*119*) on the separation of the turbulent incompressible layer. An interesting experimental study is that of Bursnall and Loftin (*128*) on localized regions of laminar separation, which has been continued by Gault (*129*).

Hypersonic boundary layers account for some 15 papers, and this is a field of growing interest.

Stability and transition form the principal topic of about 70 papers, with special attention in recent years to the compressible boundary layer and the effects of heating and cooling. The effects of roughness and wall waviness on transition are of renewed interest.

Hammerlin, a student of Görtler, has recomputed (*130*) the stability limits for concave walls and confirmed Görtler's old result as correct, although a new computation by Meksyn (*131*) had given somewhat different results. New experiments have been made on the nature of transition by Emmons (*132*) and by Evvard (*133*) and his colleagues. Dunn and Lin (*134*) have reopened the question of the stabilizing effect of cooling by showing that compressible boundary layers may be more sensitive to three-dimensional disturbances than to the two-dimensional disturbances for which the calculations have been made.

About 70 papers deal with the more specific applications of the boundary-layer concept to the heat-transfer problem. The work of Tifford and his coworkers (*135*) on laminar heat transfer deserves special mention, as does that of R. A. Seban, S. Levy, and their colleagues (*136*)

Control of the boundary layer by suction is still a very active subject, with some 40 reviews. Area suction and suction through slots for control of separation and for drag reduction are represented.

There are many papers that do not fit the categories selected for discussion, about 35 in number. I shall mention only the beautiful experiments of Dhawan (*137*) on the direct measurement of skin friction and the studies of the general mathematical character of solutions of the Navier-Stokes equations, with special reference to boundary layers and shock waves, that have been carried out by Lagerstrom and his colleagues (*138*) at California Institute of Technology.

Most of the work referred to previously dealt with boundary layers in two-dimensional flow. There are about 10 reviews on axisymmetric boundary layers.

The more difficult study of flow in three-dimensional boundary layers was begun by Prandtl (*139*) in 1945. I shall not discuss this subject, represented by about 40 papers, since W. R. Sears has recently reviewed this field in an article (*140*) in *Applied Mechanics Reviews,* except to mention the important Mangler transformation published in 1948 (*141*). This makes possible the determination of a two-dimensional boundary-layer flow that is equivalent to a given axially symmetric boundary-layer flow. It is valid both for compressible and incompressible boundary layers.

There are some 10 reviews dealing with boundary layers on spinning bodies. There are also a few papers on boundary layers with a gas or liquid injected into the layer through a porous surface. There are about 15 papers on boundary layers in oscillating or other nonstationary flow.

If one examines the statistical picture of the whole 50 years, there are of the order of 800 papers written by 500 authors. Half of the papers were written by less than 100 of the authors. About one-third of these highly productive authors are experimentalists, the other two-thirds deal with theory. The theoretical workers write about 1.5 times as many papers per worker as the experimentalists. The current total rate of production of papers is about 10 papers per month, nearly 9 times the rate immediately preceding World War II. Such rough statistics are interesting but may be misleading. Papers such as those on the Tollmien-Schlichting theory of the stability of the laminar boundary layer and on the Schubauer-Skramstad experimental confirmation are very rare indeed.

The growth of interest in the new concept of Prandtl during the first 30 years was very slow, partly because of slow communication to groups in other countries and partly because there was no direct experimental evidence of its correctness. Measurements of flow within the boundary layer had to await the development of hot-wire anemometers and very small pressure probes. There was, however, a more important factor in the initial slow growth. Early papers were mainly doctoral dissertations, and many of the authors turned to other fields after receiving their degrees. There was no engineering or technologic activity in industry to employ them to apply or to extend their knowledge of boundary-layer flow. Only when the aeronautical industry reached a relatively advanced state of engineering development were practical applications found. Then the boundary-layer concept was fruitful, not only for computation of skin friction and flow separation, but also for unifying and clarifying experimental data from wind-tunnel and flight experiments. The expansion of aeronautics stimulated basic research; it also provided a demand for engineers and scientists familiar with theoretical and experimental aspects of flow in boundary layers.

Along with successful application in aeronautical engineering, the boundary-layer concept diffused into other engineering fields—mechanical engineering, hydraulic engineering, and chemical engineering. Similarly, studies of heat transfer, diffusion, and evaporation in moving fluids were greatly aided by knowledge of boundary-layer flow. These developments interacted to produce an almost exponential growth of interest in research on boundary-layer flow within the past decade. The progress of the new concept has been very great, and the intense activity of the present leads us to hope for further advances in understanding and application in the near future.

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