SCIENCE

Vol. 102 FRIDAY, SEPTEMBER 28, 1945 No. 2648

Science: A Weekly Journal, since 1900 the official organ of the American Association for the Advancement of Science. Published by the American Association for the Advancement of Science every Friday at Lancaster, Pennsylvania.

Editors: Josephine Owen Cattell and Jaques Cattell.

Policy Committee: Malcolm H. Soule, Roger Adams and Walter R. Miles.

Advertising Manager: Theo. J. Christensen.

Communications relative to articles offered for publication should be addressed to Editors of Science, The Science Press, Lancaster, Pa.

Communications relative to advertising should be addressed to Theo. Christensen, Advertising Manager, Smithsonian Institution Building, Washington 25, D. C.

Communications relative to membership in the Association and to all matters of business of the Association should be addressed to the Permanent Secretary, A.A.A.S., Smithsonian Institution Building, Washington 25, D. C.

Annual subscription, $6.00 Single copies, 15 cents

ON THE PROBLEM OF APPLIED MATHEMATICS

By Dr. JAMES H. TAYLOR

THE GEORGE WASHINGTON UNIVERSITY

Almost anything is tremendously complicated. Frazier set out to write a monograph on a certain cult of ancient Greece and ended up with his famous work of twelve volumes entitled "The Golden Bough." Consider a simple hobby like stamp collecting. One is soon involved in a maze of pertinent details such as kinds of paper, methods of printing, types of perforations, colors and shades which even the Bureau of Standards might find difficult or impossible to determine, and finally watermarks, overprints, surcharges and forgeries.

In comparison with any other well-recognized body of knowledge, mathematics must be relatively simple. Consider for instance a given mathematics, that is, a branch of mathematics like Euclidean geometry or projective geometry or real variable. You perhaps know how any one of these subjects may be set up. After two thousand years of experience the "natural" way appears to be as follows. One lists a set of undefined elements and relations, and a set of unproved propositions involving them; and from these all other propositions or *theorems* are to be obtained by the methods of formal, deductive logic. The unproved propositions which are imposed are called axioms, postulates or assumptions. For example, in projective or in Euclidean geometry "point" and "line" are undefined elements; the relation "on" is an undefined relation. One of the axioms reads as follows: If A and B are distinct points there is at least one line on both A and B.

It is important to appreciate that a mathematics seems to be completely determined once the postulates have been listed. One's skill in the field is measured by his ability to rearrange the statements implied by the postulates so as to exhibit other interesting relations. In analytic geometry a student is given the equation of a certain circle. Of course all the properties of the particular circle are incorporated in the coefficients of the equation. The student thinks "buried" is a better word. At any rate the problem is to extract from the coefficients the information asked for. The development of a mathematics from the postulates is closely paralleled by a variety of solitaire called idiot's Delight. In this game all the cards are exposed; therefore all the possibilities are revealed once the tableau is laid out.

A postulational setup for a branch of mathematics is the complete "blue print" for the whole subject. In Veblen and Young's "Projective Geometry" this postulational material requires perhaps not more than twenty pages for its presentation; the remaining eight hundred pages are given over to the development of the theorems. The complete structure may be regarded as a "model" constructed according to the blue print. The construction is carried out by the processes of formal logic. I do not feel competent to make any statements regarding the methods of logic. However, these procedures seem to be extremely reliable; they seem to be as dependable as any so-called "law of nature." Some physicist, I believe, has called attention to the slight probability that one hundred monkeys seated before one hundred typewriters would each write the preamble to the Constitution of the United States. Personally, I think the probability that this should happen is about the same as that one hundred competent persons starting with the same system of postulates should fail to arrive at mathematics which were consistent with each other.

That the initial set of assumptions should be consistent, that is, not self-contradictory, is of course a logical necessity. This brings us to our first observation concerning applied mathematics. By "applied mathematics" we shall mean a correspondence between the elements and relations of a mathematical system and those of some other system such as a physical system or a social system if that is possible. The example we have in mind at this stage, however, is an instance of "applied physics"; that is, physics applied to the mathematics. For this is the method by which we test the logical consistency of a system of postulates. Such a system is accepted as being consistent if, when the undefined elements and relations are identified with observed elements and relations of a physical system, the postulates are observed to hold in the physical system. Here too, "seeing is believing." Just recently some of you heard Professor Von Neumann discuss the consistency of a certain logical system by an appeal to the laws of quantum mechanics. I well recall that following a lecture by Professor Veblen a number of years ago some one made the observation that "physics seemed to be becoming mathematics." In reply Veblen said that if he were crowded a bit he would have to admit that "mathematics is physics."

After a given mathematics has been examined in this manner and "certified" it may, of course, be used to test the validity of other branches of mathematics. Thus the consistency of a system of postulates for a geometry may very well be established by means of the properties of the real number system.

When a mathematics has become sufficiently advanced to go over to a postulational formulation, it has also become quite artistic in nature. Aesthetic considerations require that the fundamental assumptions shall be independent among themselves; that is, no one of them shall be a logical consequence of the remaining ones. Here again, we may employ an empirical test. One merely looks about and attempts to observe a set of objects with certain relations between them, and such that if they are identified with the undefined elements and undefined relations of the mathematical scheme, it is observed that all the postulates except one are satisfied and it is contradicted. If such a concrete representation can be observed we accept its mere existence in the physical world as conclusive that the violated postulate is not a logical consequence of the others. Such a test may be easily applied in a given case, and on the other hand it may be very difficult to find a suitable representation. It required about two thousand years to conclude that the parallel axiom of Euclid is really independent of the remaining postulates of the Euclidean geometry. You will recall this assumption states: Through a given point there exists one and only one line which is parallel to a given line. Scholars in the field believe that Euclid himself was concerned with the possible redundancy of this postulate. Such concern indicates that the artistic structure of the subject was well appreciated even in his time. Naturally the frontal attack on the problem was to attempt a proof of the axiom as a theorem derived from the remaining axioms. This turned out to be one of the great classical problems of mathematics. The problem had a simple formulation and the answer was not known. It was the kind of a problem an "amateur" might solve and thereby become famous.

A little over one hundred years ago the problem was finally disposed of by the technique of the "reducto ad absurdum" proof. This was simply to accept all the other postulates of Euclidean geometry and to deny the parallel postulate; then proceed to develop the resulting subject. If a contradiction appeared it would be concluded that the assumptions made were inconsistent, and hence that the parallel assumption was really a consequence of the others since its denial led to a contradiction in the system. However, no contradiction appeared and thus there arose a non-euclidean geometry. This was the first instance of an abstract geometry developed quite independently of the physical world. It was like a new synthetic sugar created in a chemist's test-tube. Its production marked a mathematician as being more than merely a recording engineer.

This emergence of a fully fledged abstract geometry in no wise related to the world of experience as then known was an amazing event. It could not have appeared much sooner in the time scale. In the first place its development depended upon a certain technique in abstract mathematics which required centuries to establish, and in the second place it probably would have been rejected by an earlier generation because of its conflict with their intuitive notions. With the advent of relativity theory we see that the non-euclidean geometry may very well have a concrete representation in the physical world.

I would guess that the art of painting was initially merely a recording process. It must have been a long time before any philosophy of painting came into being. However, later on the painter attempted to portray conceptual objects, whereas in the early stage the effort was simply to record perceptual objects. Mathematics has no doubt followed a similar life pattern. At first the symbols 1, 2, 3, etc., were names of an ordered sequence in close association with sets of perceptual objects. The symbols were made to correspond with perceptual objects, and operations or relations involving the symbols were introduced which were in correspondence with certain observed physical processes. A correspondence which is preserved under the corresponding operations is called an isomorphism.* This notion of isomorphism is inherent is every problem of applied mathematics.

After the rules of the "arithmetic" had been formulated in conformity with observed physical processes the mathematics became something more than merely a picture. It had acquired a structure, and in order to lend meaning in all cases when the rules were applied to the symbols, new symbols or numbers were added. Thus certain elements in the mathematics may well be purely conceptual and not have their counterpart, that is, corresponding elements in the perceptual world. These conceptual elements are likely to bring about closure under the operations; they give to the theory a coherence or continuity which it would not otherwise have. These conceptual elements, like the irrational numbers, behave like a gelatinous material which holds in suspension the elements which correspond to the perceptual objects. The rational numbers are the numbers which correspond to observed measurements. No carpenter, engineer, or scientist ever measured a dimension or read any dial or scale except as a rational number. However, since any irrational number can be approximately arbitrarily closely by a rational number, so far as empirical evidence goes, the test of the correspondence of the Pythagorean theorem for a right triangle whose sides are each of unit length with any physical triangle of such measured dimensions is just as conclusive as if the square root of two were actually a rational number.

When a mathematical model has been selected to represent a certain class of phenomena it is likely that some interesting aspects of the physical model will not at first seem to have their images revealed in the mathematical model, and conversely. Thus mathematics and a field of application mutually support one another, at least until contradictions begin to appear. By a contradiction we mean an instance in which a theoretically computed result fails to agree with the corresponding observed result by an amount larger than the allowable (experimental) error. Since the mathematics is assumed to be consistent, and the physical system is likewise assumed to be consistent, a contradiction of this sort is accepted as conclusive evidence that the hoped-for isomorphism is not really valid. Even though the initial fit of the mathematical and physical models may appear to be very good, the normal case history is to the effect that there will come a time when the discrepancies of the two models can be no longer ignored. When this occurs, modification is called for, and an adjustment may bring about order. Unless such an adjustment is possible the given subject ceases to be one to which the given mathematics can be applied. Moreover, such a breakdown of the isomorphism throws doubt on all the happy correspondences of the past. From a "practical" standpoint this renunciation may not be disturbing but from a philosophical point of view it is; for at heart mathematics and science are as dramatic and tragic as an Ibsen play.

Newtonian dynamics and the mechanics of rigid bodies are probably the most conspicuous successes of applied mathematics. Newtonian mechanics seemed perfect for over two hundred years. It is still sufficiently good to enable you to predict the time of a total eclipse of the sun a hundred years or so hence to within an error of a fraction of a minute, and the boundaries of the path of totality to within a few city blocks. However, it is not good enough to account for an advance in the perihelion of Mercury of about 42 seconds per century. We note that this discrepancy is one that required time in which to make its appearance; it is an accumulative difference. However, because of this property it became increasingly insistent for recognition. The easing of this contradiction is generally cited as one of the triumphs of relativity theory. Despite the contradiction mentioned, the construction engineer will continue to employ Euclidean geometry because of its relative simplicity, and not the geometry of relativity theory.

If a contradiction which arises between the mathematical and physical models can not be removed by a reasonable adjustment in one or the other or both, it may be decided that the departures, even though noticeable, are still statistically significant. Whether or not this is the case is probably largely a matter of opinion. If the departures become too great even statistics is no longer interested.

The extreme case in the other direction is when the two models fit perfectly, when there is no longer any doubt about the isomorphism. The complete establishment of the isomorphism automatically transforms the applied mathematics into pure mathematics. Some people object to the term "pure mathematics." Naturally the connotation is quite different from that in chemistry, for example. Here the term is descriptive of a lack of refinement inasmuch as no effort is made to interpret the symbols; the theory has structure but no substance. The observable world is the answer book for applied mathematics; pure mathematics has no answer book. Evidently bookkeeping is pure mathematics. A bookkeeper in a brokerage firm need not know what the symbols stand for; he merely needs to know a few simple rules of abstract arithmetic. In earlier stages theoretically mechanics was properly regarded as applied mathematics. However, this is no longer the case, for theoretically mechanics under well-stipulated laws is a deductive science and has precisely the characteristics of an abstract mathematics. Having different names for certain symbols, such as "particle" in place of "point" or "velocity vector" in place of "tangent vector," is quite analogous to substituting the appropriate words of another language for these objects. There is of course a field of applied mechanics and the mathematics which is likely to be applicable is that of theoretical mechanics. A parallel situation occurs in optics. Thus there is a mathematical subject called geometrical optics and an associated one called physical optics. It is interesting to note that the recently established Westinghouse Fellowships in Electron Optics are available only to those with specialized training in the field of physics, mathematics or electrical engineering.

A measure of the "logical" qualities of a subject is the uniformity of the results which different individuals obtain from a given initial set-up. By this test mathematics is perhaps the most logical body of knowledge in existence. I am of the opinion that one of the contributing reasons for this is the fact that the mathematician does not know what he is talking about; that is, he has no concrete interpretations for the symbols. This being the case he has no personal interest in the outcome. He is not under a certain nervous tension as may be the case when adding the bills due during the current month or when making out the income tax return. The above observation concerning the logical aspects of mathematics must not be confused with a prevalent but erroneous one, to the effect that a mathematician must be a very logical person. On the contrary, he is likely to be less logical than the average. The correct statement is: The mathematics which a mathematician does is logical. In a great deal of his work he has learned to rely upon the symbols and their admissible relations, and he is therefore handicapped when attempting to deal with situations not expressed in terms of the symbols with which he customarily works.

If a system is to be amenable to mathematical treatment it must have two properties: (1) It must be a logical system, and (2) in some sense it must be extremely simple. Hobson ("The Domain of Natural Science") writes: "Number and extension, with which Arithmetic in the extended sense and geometry, are alone directly concerned, can be developed as sciences descriptive of only the most superficial aspects of the perceptual world, since they leave aside almost all the properties and qualities of perceptual things as irrelevant for the purpose of their construction, so that in these departments the process of abstraction and idealization can take gigantic strides at a very early period in their genesis."

If all the properties of an object are taken into consideration, it is clear that no observable object is simple. To achieve simplicity all properties of the object not deemed essential for the particular purpose are supposed "stripped off." To make effective progress it must be possible to isolate the system from the rest of the universe. So far as Newtonian dynamics is concerned the solar system is effectively isolated. As knowledge of such possibly isolated systems becomes better and better organized we may expect mathematics to play an increasing role in their further development. Mathematics has already established itself as being an important adjunct in almost any investigation in the physical sciences. The biological sciences are entering a phase in which mathematics may prove useful, although I suspect the biologists will have to be content with statistical procedures.

When we attempt to apply mathematics to a social system, such as might appear in the subject of sociology or economics, grave difficulties arise. For such a system is not logical, and it is not simple. There is no reason to suppose that a rigorously logical structure will fit any region of human activity where events, as they appear in the time scale, have no predictable, or even expected, uniformity. Consider this question: Why is the salary of a judge of the Supreme Court not subject to Federal income tax? After you have answered this by a sequence of logical moves try another one: Why should a responsible Government sponsor a program which calls for slaughtering of pigs in Iowa, burning of wheat in Kansas, support of a subsidy for decreased agricultural production, and at the same time construct a Boulder dam, one of the reasons for such construction being to open up several hundred thousand acres of tillable land? The acts and decisions of people are very largely unpredictable, and it is this uncertainty, I believe, which almost precludes the possibility of applying mathematics, in any practical sense, to a social system.

In particular, there has been a noticeable lack of satisfactory results from attempts to apply mathematics to certain aspects of economics. I have just offered my explanation of these failures, but some other reasons have been advanced which still hold out a little hope. One of these is: there are variables present which are not even recognized, and of those variables which are admitted, we do not know how a variation of a given one will change the functional value of the function we seek to determine. This argument is based upon the tacit assumption that the mathematics is adequate and that the difficulty is due to the complexity of the situation. There is no doubt but that it is very complex. Take the simplest economic problem which you can conjure up and contrast it with the problem of formulating Newton's Law of Gravitation. In the latter case you suppose two bodies with masses m and m' respectively, at a distance r apart supposed relatively large with respect to the maximum diameter of either body. We want a formula which gives the force F of attraction between the bodies; that is, we are looking for a function of the variables m, m' and r which yields the proper functional value when numerical values are inserted for the variables. One recognizes that there must be involved a constant k depending upon the units in which the quantities are measured, and that the function is probably symmetric in the variables m and m' and that it is properly monotonic decreasing in r. Naturally there are infinitely many functions with these properties. Of all the functions the desirable one, if any, is the simplest one which yields results within the tolerances allowed. One naturally tries kmm'/r but that is no good. The next guess is better, for this time we take the denominator to be r^2 instead of r. The whole problem is perhaps not quite so simple as we have just indicated. However the point of emphasis here is that the problem is relatively simple in comparison with a corresponding problem dealing with any social system.

Von Neumann and Morgenstern in their amazing book, "Theory of Games and Economic Behavior," offer another reason for the failure of mathematics to achieve significant results in the economic field; namely, that the mathematics used has not been adequate to analyze such phenomena. They call attention to the fact that most of the mathematics has been developed for the purposes of physical science. The theory of the real number system, or linear continuum, then naturally occupies a conspicuous part in most of the mathematics. One of the important properties of the real number system is that it is a well-ordered set. That is, given two real numbers, x and y say, they satisfy one of the three order relations, x is greater than y, equal to y, or less than y. Moreover, this relation is transitive. That is, if x is greater than y, and y is greater than z, then x is greater than z. Also, if a is greater than b, and c is greater than d, it follows that the sum a plus c is greater than the sum b plus d. This relation may, however, not hold in an economic theory. For it may be that of two possibilities, A will be preferred to B, and of two other possibilities C will be preferred to D, but that A and C will not be preferred to B and D. Imagine the average school teacher retiring at age sixty-five. He will probably prefer a life annuity, under the relatively favorable terms which existed a few years ago, which is purchasable for twenty-five thousand dollars to a cash settlement of that amount. In the hypothetical case in which he is confronted with two such decisions he will probably elect to receive the fifty thousand dollars in one sum.

Instead of attempting to analyze economic situations, as such, Von Neumann and Morgenstern content themselves with mathematical aspects of certain games. The idea is that these games present to some extent the structure that might be expected in more serious economic "games." The book is largely given over to existence theorems, and the complexity of the mathematics is truly prodigious. Not many persons in the world are able to write such a book, and it is my opinion that there are still fewer economists in the world who are able to read it.

It may turn out that the biologist can be of more use to the economist than the mathematician. I have a friend who is interested in correlating sunspot activity with prices of the stock market. He seemed surprised when I asked why these things should have anything to do with each other. "It's quite simple," he said, "you see, during periods of high sunspot activity people are more optimistic in their outlook than usual, and hence are inclined to take more chances."

OBITUARY

FLORENCE BASCOM
1862–1945

THE first and probably the foremost American woman geologist, Dr. Florence Bascom, died on June 18th in Williamstown, Massachusetts. She was a fellow and former vice-president of the Geological Society of America, professor emeritus of Bryn Mawr College and a retired member of the Petrological Section of the U. S. Geological Survey.

Miss Bascom was born in Williamstown in 1862, the daughter of the late John and Emma Curtiss Bascom. Her father was professor of philosophy in Williams College, and her early years were typical of the New England of her time. A strict but intellectual family life in a New England college town may have, in large measure, served to develop the integrity of thought and intellectual honesty which were notably characteristic of her in later life.

Her father later became president of Wisconsin University. Florence Bascom entered that university, graduating in 1882. Up to that time her interests had not been notably scientific. She was a brilliant student, mastering many subjects, with a keen interest in all fields of knowledge. She continued to study at the University of Wisconsin, taking an M.A. in 1884. By that time she had definitely turned to geology, especially the almost unknown field of petrography. In the eighteen eighties the microscopic study of minerals and rocks was in its infancy. Most of the pioneer work in this line was done in Germany, and the original German papers were the only descriptive literature. Those of us who are old enough to have begun our study of petrography before Rosenbusch's text-book had been translated can testify as to the difficulty of mastering the optical properties of minerals from a German description. In her case there was not even a text-book. That she did master the subject and decide to make it her specialty is an indication of her courage and of that determination which found nothing too difficult.

When Johns Hopkins University opened its Graduate School to women, Miss Bascom entered and continued her petrological studies under the late George H. Williams. Her thesis was on certain formations in the South Mountain in Maryland. These formations had previously been regarded as sediments, but her study of them under the microscope proved them to be altered volcanics. She named them "Aporhyolites," the prefix "apo" long remaining in general use for igneous rocks altered by recrystallization. Miss Bascom received her Ph.D. from Johns Hopkins in 1893. She was the first woman to receive that degree from Johns Hopkins.

For the next two years she was instructor and associate professor of geology at Ohio State University. In 1895 she went to Bryn Mawr College, and there founded the Department of Geology. It is probable that her prestige as a research scientist was responsible for this opportunity. Bryn Mawr has, from its beginning, put a high value on productive scholarship, and Miss Bascom was of great promise in that direction. There was no intention of establishing a new important scientific department. Geology was thought to be a subject that would have no wide appeal to women, and the college lacked both funds and space. Dalton Hall, the science building, had recently been completed and its three main floors were already fully occupied by the established departments of physics, chemistry and biology. A fourth floor was used for storage by these three departments. It was on this fourth floor that a small office was boarded off for Miss Bascom, and a slightly larger room for combined lecture room and laboratory. This classroom was arranged for twelve students, and that was the size that one elementary class was expected to remain.

While Miss Bascom fully agreed with the tradition of the college in regard to research, her ideas were quite different as to the importance of geology in a college. She felt that geology was the most important of the sciences, to some extent including the others, and that it was really the culmination of all science. Probably no one will ever know all the difficulties that she encountered, but little by little she achieved her purpose of making her department one of the best in the country. In 1899 she was teaching petrography, having acquired one really good petrographic microscope and an admirable collection of thin sections of minerals and rocks. In 1906 she became full professor, and had an associate who relieved her of