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ON THE TEACHING OF MATHEMATICS[1]

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THE principal obligation incurred by a vice-president of the American Association for the Advancement of Science upon his election to office is the preparation, and deliverance before the section of which he is chairman, of an address. In discharging this obligation I wish to speak on a subject which I regard as of fundamental importance, namely, the teaching of mathematics. By this I mean not merely the provision of information about mathematical methods and results but also the development of an interest in, and understanding of, the spirit of mathematics. For I take it as evident that no teaching can be successful which attempts to skim off the products of mathematical fermentation and ignores the process of fermentation itself.

It is unnecessary for me to dwell on the fact that the demands of war have focussed a strong searchlight upon the mathematical capabilities of graduates of our high schools and colleges, nor to call to your attention the fact that the disclosures are disquieting. All of us who teach know that it is possible for a young man to spend twelve years in school and yet not know with that assurance, which comes only from a thorough understanding, how to add fractions. When we meet a young man in college who calculates thus:

$$\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$$

it is not enough to chide him for his stupidity. No, the fault lies in the manner of his teaching. We expect our students to add $\frac{1}{2}$ to $\frac{1}{3}$ and obtain a result without knowing what $\frac{1}{2}$ or $\frac{1}{3}$ is. I visualize you as objecting thus: "Why, this is absurd; every young student is told that $\frac{1}{2}$ is one half, *i.e.*, one divided by

[1] Retiring vice-presidential address before Section A of the American Association for the Advancement of Science, Cleveland, September 12, 1944.

2." The obvious rejoinder to this is: if one can be divided by 2, why not tell the answer? Why write both the dividend (1) and the divisor (2), when giving the "answer" $\frac{1}{2}$, save for the fact that the division is impossible? The simple truth is that fractions were introduced just because the division of one whole number by another is not always, although it is sometimes, possible. On sober thought the wonder is that so many people succeed in learning how to add fractions correctly without ever knowing clearly what a fraction is; not that so many people fail to learn how to add fractions. This is, then, the burden of my address; I believe that our failure, in so far as we have failed, lies in our hurry to secure results without understanding the processes involved. I believe further that our haste is vain; that the time necessary to gain the understanding is more than made up for by the increased rapidity with which the results can be mastered after the understanding has been gained. I believe also that those who fail to gain, even when properly taught, the understanding for which we should strive become at least as technically proficient as those who are trained to perform in a routine manner calculations whose significance they can not understand.

Since criticism is valueless if it does not prove itself constructive, I shall indicate briefly how I think mathematics should be taught. Starting, as every one does, with the counting numbers, I would emphasize their two essential features, namely, the existence of a leader 1 and the existence of a follower for each number, and I would point out that these imply the sensational and somewhat disquieting fact that whilst there is a leader or beginner, there is no ender. I would train my students to distinguish between definitions and theorems. For instance, if I ask what is the number twenty, I do not want to be told that it is twice ten; for I do not know (at the beginning stage) what twice means, and I may just as well ask what ten is as what twenty is. The answer I expect is: Twenty is the follower of nineteen. I see at once that a difficulty confronts me; this answer is completely satisfactory if I am lucky enough to know already what nineteen means, but if I do not know this, it is of no use to me. But a door opens! I can now push back to the leader. Nineteen is the follower of eighteen; eighteen is the follower of seventeen and so back to one. But what is one? Do not spoil it all by a false sophistication and say that one is the follower of zero! No, we must be honest; one is simply the beginner or leader of the counting numbers. Being a leader he does not follow any one.

The stage is now set for the process of addition and the learning of the addition table. How does one know that "two and two is four"? Many of my friends and students regard this as a trick question; the statement being so self-evident as not to permit demonstration. I then ask what does "two and two" mean, and, amazing as you may think this to be, I seldom receive a clear-cut answer. Of course everything rests on the sturdy shoulders of the leader; "and one" means merely "the follower." "And two" means "the follower of the follower" and so on. Thus two and two is the follower of the follower of two, *i.e.*, the follower of three, *i.e.*, four. What is meant by nine and seven, *i.e.*, $9+7$? and what is meant by seven and nine, *i.e.*, $7+9$? Are these by definition the same? These are the natural, simple questions which our present generation of students is unable to answer. Of course $9+7$ is the follower of $9+6$, whilst $7+9$ is the follower of $7+8$, and the fact that $9+7=7+9$ is a consequence of the, by no means evident, fact that $9+6=7+8$. It is clear that in order to really find out that $9+7=16$, we must push back to $9+1$ which is, by definition, 10. Even the youngest beginner should sense, at least vaguely, the universality behind the result $9+7=7+9$, which universality is indicated by the algebraic stenography: $a+b=b+a$.

Now, and only now, are we ready to explain multiplication. What is meant by two times two? Is it the same as two and two? If so, why is three times three not the same as three and three? Here again, we must turn to the leader who now, in contrast to his behavior in addition where he changes everything ($a+1$ being never the same as a), is very polite: 1 times a (*i.e.*, a multiplied by 1 which we denote by $a\times 1$) is always the same as a. What is 4×3? It is $(4\times 2)+4$. And what is (4×2)? It is $(4\times 1)+4$, *i.e.*, $4+4$, *i.e.*, 8. Thus multiplication is defined in terms of addition. $2\times 2=2+2$ merely because $(2\times 2) = (2\times 1)+2$, and, due to the politeness of the leader, $2\times 1=2$. Is it, then, remarkable that $6\times 9=9\times 6$, or is this an obvious consequence of the definition? Well, $6\times 9=(6\times 8)+6$, whilst $9\times 6=(9\times 5)+9$, and it is quite remarkable that $48+6=45+9$. Nevertheless, the beginner should be told that this remarkable fact may be proved, and the universality of the law: $a\times b=b\times a$ should be pointed out. But our intelligence must protest against such a demonstration as the following: to prove that $6\times 9=9\times 6$, construct a rectangle of sides 6 and 9. Then the area of the rectangle is at once 6×9 and 9×6; hence $6\times 9=9\times 6$. If you tell me such things I have to ask you what you mean by the area of a rectangle. If you answer that you mean the product of the two sides, I have to ask you which product? If you assure me that it makes no difference and advise me not to worry, I may trust you, but we are not discussing mathematics. Faith is a very important thing, but it can not be taught.

Every schoolboy learns the multiplication table as far as 10 times 10. No single item in his later study of mathematics is as difficult or makes as many demands upon his memory. At this stage in his study, interest in mathematics may be aroused by the mere statement of the following rule which seems to be novel to most of my friends: to multiply two numbers which lie between 5 and 10, add their excesses over 5 to obtain the tens figure and multiply their shortages under 10 to obtain the units figure. Thus $7 \times 9 = 63$, since $2 + 4 = 6$ and $3 \times 1 = 3$.

We have now reached a critical point in the teaching of mathematics. A high percentage of educated persons report somewhat as follows: "I was always rather good at arithmetic and enjoyed it; but I never did understand those negative numbers and why one has to be careful in working with zero. These things floored me and I have had ever since an inferiority complex with respect to mathematics!" To understand what negative numbers and zero are, one must change entirely one's attitude towards number. It is no longer sufficient to count; we must count from somewhere. If we visualize the numbers as men in a parade we are no longer concerned with where in the parade is a particular friend Tom of ours. We have two friends, Tom and Dick, and what concerns us is their relative position: where is Dick compared with Tom? Does he come earlier or later and how far apart are our two friends? The thing we call a number now is a pair (Tom and Dick) of counting numbers, *e.g.*, 4 and 7, where it is of the highest importance which is Tom and which is Dick. In other words, we concern ourselves with an ordered pair, *e.g.*, (4, 7) of counting numbers, and we call this pair *a* number (*not* two numbers). To justify, however, the name number, we must learn how to add and how to multiply these ordered pairs of counting numbers. To bring clearly into focus the fact that what really concerns us is the relative position of Tom and Dick, we recognize that the relative position of Tom and Dick is the same as the relative position of the follower of Tom and the follower of Dick. Thus, for instance, we have to agree that (4,7) is the same as (5,8) and this again is the same as (6,9) and so on. Thus the *same* number may have *many* appearances or costumes. How can we penetrate this disguise? The answer is very simple: (4,7) is the same as (5,8) because $4 + 8 = 7 + 5$; and, in general, (a,b) is the *same* as (c,d) when, and only when, $a + d = b + c$. This being agreed on, a serious political situation confronts us. On one side we have the early settlers, the honest-to-goodness numbers, the counting numbers which constitute a kind of aristocracy amongst the new comers—these ordered pairs of counting numbers which have so many costumes which they change at will and without notice. This class distinction would be fatal to any convenient mathematical theory, and we abolish it. We decree that the leader, 1, of the aristocrats must dress up in any one of the costumes (1, 2), or (2,3) or (3,4) and so on. His follower 2 must wear one of the costumes (1,3), (2,4), . . . , (6,8) and so on. In general the counting number n must dress up as $(1, n+1)$ or $(2, n+2)$ and so on. This decree is absolute; if any one of the counting numbers refuses to wear the costume assigned him, we do not admit him to the *new* republic of numbers; he must continue to live in the country or republic of counting numbers. If he wishes to do business with the republic of ordered pairs, he must deal through his representative who must wear the prescribed costume.

Having now agreed upon the appearance of, or costume worn by, our numbers, we must turn to the really important problem of writing their constitution; in other words, we must lay down the laws of addition and multiplication under which they must live. You must understand clearly that one can not prove these laws; they merely express what we regard as right and proper. Mathematics being a very pragmatic science, the words right and proper are merely synonyms for the word convenient; and of all revolutions the most convenient one is a bloodless one. We assure our aristocrats, the counting numbers, that, when they assume the toga of citizenship in the republic of "ordered pairs," they will not have to change in even the slightest manner their accustomed mode of life. The laws of addition and multiplication of ordered pairs are so framed that "and one" will still be the (dressed-up) follower; and, further, the leader 1 will still be polite in multiplication. What are these laws of addition and multiplication? The law of addition is extraordinarily simple; merely add the first parts of each pair and the second parts of each pair, separately, to obtain, respectively, the first and second parts of the sum:

$$(a,b) + (c,d) = (a+c, b+d).$$

To check that $3 + 4$ is still 7, we write 3 as (1,4) and 4 as (1,5) and find that $(1,4) + (1,5) = (2,9) = (1,8)$ (since $2 + 8 = 9 + 1$) and (1,8) is merely one of the costumes worn by 7. Even the beginner may grasp the remarkable fact that this law of addition depends in no way on the costumes worn by the individual terms. The law of multiplication is much more complicated:

$$(a,b) \times (c,d) = (ad + bc, bd + ac).$$

In words: use cross multiplication (*i.e.*, a with d, and b with c) and addition for the first part, and *straight* multiplication (*i.e.*, a with c and b with d) and addition for the second part. This complicated rule of multiplication may be remembered as follows: first multiply each number of the first pair by the second number of the second pair obtaining (ad,bd); then multiply each number of the first pair by the first number of the second pair and reverse the order obtaining (bc, ac); finally add the two pairs (ad, bd) and (bc, ac) to obtain the product $(ad+bc, bd+ac)$. To verify the politeness of 1 when wearing, for example, the costume (3, 4), let us multiply 6, wearing the costume (4, 10), by (3, 4); we obtain

$$(4,10) \times (3,4) = (16+30, 12+40) = (46,52)$$

which is one of the costumes of 6.

The whole point of this bloodless revolution is that the franchise is extended to a great body of new citizens; if it merely meant that the counting numbers were to dress up and carry on their ordinary mode of life, it would be play acting and unworthy of serious attention. Amongst the new citizens is a very remarkable one whose costume is distinguished by the fact that both of its parts are the same; for instance (1,1),(2,2), . . . , (9,9), . . . are several of these costumes. The amazing characteristic of this man is his dual personality. He is (like the leader 1 of the counting numbers when multiplication is being performed) utterly polite when addition is being performed:

$$(a,b) + (1,1) = (a+1,\ b+1) = (a,b)$$

(simply because $(a+1)+b = (b+1)+a$). To make up for this politeness (almost a lack of interest) when addition is being performed, he is utterly ferocious (a bandit really), when multiplication is being performed: the product of any number whatsoever by (1,1) is (1,1). For example,

$$(8,15) \times (1,1) = (8+15, 8+15) = (23, 23) = (1,1).$$

This bandit-like behavior is so characteristic that other numbers fight shy of (1,1) in the following sense: if we know that the product of any two numbers is (1,1), we may rest assured that one of these two numbers was (1,1) itself. This is the fundamental principle underlying the solution of all algebraic equations. We simply maneuver some statement, or combination of statements, concerning numbers into the form of statement that the product of two numbers (one or both usually unknown) is (1,1). This solves, at least partially, the mystery: one or other (which one we do not know) of the two unknown numbers is (1,1).

This new number is the number zero. What are the negative numbers? The negative of a number is the number obtained by interchanging its two parts. Thus -5 (the negative of 5) has (6,1) as one of its costumes simply because (1,6) is one of the costumes of 5. It follows at once that the negative of zero is zero, and that this is a characteristic property of zero; no number other than zero is its own negative. Furthermore, the sum of any number and its negative is zero; for instance $(4,9)+(9,4) = (13,13) = (1,1)$. It is easy to see that the negative of any number is the product of that number by $(2,1) = -1$. If we imagine each ordered pair as realized by a coin with one of the two parts of the number on each of its sides, we obtain the negative of a number by turning the coin over. The fact that a double turn over is as if the coin were not turned over at all finds its expression in the mysterious Rule of Signs:

$$- \times - = +.$$

It should be now clear that such a question as: Can you have zero apples? or -5 apples? merely betrays a complete lack of understanding of what zero or -5 is. Neither of these is a counting number and so one can not count with them; I can not have zero apples, but I can have the same number of apples as you; similarly, I can not have -5 apples, but you can have five more apples than I have. The numbers which express my standing relative to yours with respect to the possession of apples, are, respectively, zero and -5.

We have seen that we have been able to achieve a bloodless revolution, but it has not been an inexpensive one. As part of the taxes which must be paid by us, and by posterity forever, for this revolution, is the fact that our numbers no longer have a leader; the old leader (1,2) is now the follower of (1,1) because $(1,1)+(1,2) = (1,2)$. The polite (1,1) is the follower of (2,1) because $(2,1)+(1,2) = (3,3) = (1,1)$ and so on. Each of our numbers has a follower, but there is no beginner. To those of us who are conservatively minded and who miss the leader, we can only make the consoling remark that even in the good old days, although our numbers (the counting numbers) had a beginner, they had no ender. Now the matter is at least more symmetrical; every number has a follower, but there is no beginning and no end.

Once it is clearly understood that all numbers are ordered pairs of counting numbers, it is no longer dangerous to use a condensed notation in which 5, for instance, denotes (1,6), whilst -4 denotes (5,1) and 0 denotes (1,1). The advantage of this notation is that its use renders it unnecessary to remember the complicated rule by which ordered pairs are multiplied. The ordinary multiplication table for counting numbers, together with the rule of signs and the fact that, if one factor of a product is zero, so also is the product, are all we need to know. But no real understanding of negative numbers can be gained if one thinks that the numbers we are dealing with, or at least those of them which are neither negative nor zero, are counting numbers. It is the failure of our present teaching methods to emphasize this fact that lies at the root of all the difficulty with negative numbers and zero. A further advantage which may be claimed for the teaching of negative numbers as ordered pairs of counting numbers is that the student is prepared for the concept of a vector which is so fundamental in physics and engineering. A vector simply carries one (as the name implies) from one point to another; the negative number -5 carries one from the counting number 6 to the counting number 1 whilst the positive number 5 carries one from the counting number 1 to the counting number 6.

Now what about the fractions? So that there may not be any misunderstanding, we shall refer to any one of our ordered pairs of counting numbers as an integer; thus $5=(1,6)$ is an integer, as are also $-8=(9,1)$ and $0=(1,1)$. A fraction is an ordered pair of integers which we write thus: $\frac{a}{b}$. Thus four counting numbers in all are required to define a fraction, two for the numerator a and two for the denominator b. We admit our integers (which now, in this new revolution, play the role of aristocrats) into the republic of fractions, by decreeing that each integer must appear as a fraction whose denominator is the integer 1. Thus, when we write the symbol 5 we mean the fraction $\frac{5}{1}$; this is no longer either the counting number 5 or the integer $5=(1,6)$. Just as each integer has many costumes (each a pair of counting numbers), so each fraction has many costumes (each a pair of integers). We penetrate this disguise by the rule that $\frac{a}{b}=\frac{c}{d}$ when, and only when, $ad=bc$. For example $\frac{10}{2}$, $\frac{-15}{-3}$ are appropriate costumes for the number 5.

In order to justify our calling the fraction $\frac{a}{b}$ a number we must learn how to add and how to multiply fractions. In framing these laws of addition and multiplication, we take care that they do not disturb the laws of addition and multiplication of integers; *i.e.,* we arrange that

$$\frac{a}{1}+\frac{b}{1}=\frac{a+b}{1};\ \frac{a}{1}\times\frac{b}{1}=\frac{a\times b}{1}.$$

As far as multiplication is concerned, this is very easily done; we adopt as our rule of multiplication the following:

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}.$$

In other words, we multiply the numerators of our separate fractions to obtain the numerator of our product fraction, and we multiply the denominators of our separate fractions to obtain the denominator of our product fraction.

It requires more skill to devise a rule of addition of fractions which will leave undisturbed the manner in which integers have been added. This rule is symbolized thus:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

The real trouble for the beginner is the lack of symmetry with respect to the numerator and denominator; the denominator of the sum is the product of the two denominators, whilst to obtain the numerator we cross multiply and add, *i.e.,* we multiply the numerator of either fraction by the denominator of the other and add the two products thus obtained. What, then, is the sum of $\frac{1}{2}$ and $\frac{1}{2}$? It is $\frac{2+2}{4}$, *i.e.,* $\frac{4}{4}$ or 1. I feel strongly that the accepted mode of teaching the addition of fractions with its emphasis on "the least common denominator" is responsible for most of our troubles. Even very young beginners quickly learn (and seem to enjoy) the simple rule: cross multiply and add for the numerator; multiply for the denominator. Once they learn this rule they never seem to make the mistake indicated by $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$, but, rather, say correctly $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$. If you object that the rule of addition does not give the sum "in its lowest terms" as often as does the method involving the least common denominator, I reply that this is a very small price to pay for the increase in accuracy gained. According to the rule of the least common denominator $\frac{1}{2}+\frac{1}{4}$ is $\frac{3}{4}$, whilst according to the "cross multiply and add" rule it comes out $\frac{6}{8}$, which is merely $\frac{3}{4}$ wearing another costume. But if the "cross multiply and add" rule should increase the accuracy of performance by 10 per cent. (which I believe to be a conservative estimate), it is well worth the trouble involved in noting that $\frac{3}{4}$ is a "simpler" cosume than $\frac{6}{8}$ for the desired sum. What is much more serious is the fact that our students obtain under the present methods of instruction the impression that they can prove how fractions should be added.

The difference in the behavior of the integers and the fractions with respect to the two operations of addition and multiplication should be emphasized. The addition of integers is very simple:

$$(a,b)+(c,d)=(a+c,\ b+d),$$

whilst the multiplication of integers is complicated. On the other hand, the multiplication of fractions is very simple:

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$$

whilst the addition of fractions is complicated. We may say that integers are made for addition; for them it is a natural operation which they perform without having to be taught, whilst multiplication is a skilled operation which they must learn how to perform. In contrast with this, fractions are made for multiplication, whilst addition is for them a skilled operation.

We have seen that there is a Rule of Three, namely $ad=bc$, by which we can test whether two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are the same or not (just as there is a Rule of Three, namely, $a+d=b+c$ by which we can tell whether two integers (a,b) and (c,d) are the same or not). We can use these Rules of Three to arrange our integers and fractions in order; our guiding principle being always that we leave undisturbed the order which was such a prominent characteristic of the counting numbers. We say that (c,d) follows, or is greater than, (a,b) if $a+d$ follows $b+c$. For example 0 (*i.e.,* (1,1)) is greater than every negative number; this "largeness" of the number zero should be emphasized (to counteract the impression that when you "have zero" you have "simply nothing at all," everything having vanished into thin air). In ordering the fractions we have to be a little more careful; we first note that $\frac{a}{b}$ and $\frac{-a}{-b}$ are merely different costumes for the same fraction and so we may, if we find it convenient, decree that the denominator of a fraction shall never be negative. Suppose we do this; then we say that $\frac{c}{d}$ follows, or is greater than $\frac{a}{b}$, if cb follows, or is greater than ad. But then it becomes clear that we must outlaw all fractions whose denominator is zero in view of the bandit-like behavior of zero in multiplication. The Rule of Three itself outlaws the fraction whose numerator and denominator are both zero. For it tells us that every fraction $\frac{a}{b}$ is the same as $\frac{0}{0}$ because $a\times 0=b\times 0$. If we wish to preserve the axiom of equality: "Things which are equal to the same thing are equal to each other," we would have to grant the absurdity that all fractions are equal if we admitted the fraction $\frac{0}{0}$. We outlaw then $\frac{0}{0}$. This should be emphasized; under present teaching methods too high a percentage of college students *cancel* the zeroes and say that $\frac{0}{0}=1$.

What about the fractions whose denominators are zero but whose numerators are different from zero? These are all, by the Rule of Three, equal, but no one of them is equal to a fraction whose denominator is different from zero. They must *all* be outlawed. Consider, for example $\frac{10}{0}$; this fraction follows $\frac{10}{1}$ because $10\times 1=10$ follows $0\times 10=0$. On the other hand $\frac{-10}{0}$ $\left(\text{which equals } \frac{10}{0}\right)$ is followed by $\frac{10}{1}$ because $10\times 0=0$ follows $1\times -10=-10$. Thus if we admitted $\frac{10}{0}$ into our community of fractions we would have to admit that $\frac{10}{1}$ is at once greater than and less than $\frac{10}{0}$. Why is it that the one thing that my students are quite certain about concerning fractions is that the value of $\frac{10}{0}$ is infinity? When I ask them just what they mean by the word infinity, they are usually wise enough to remain silent.

How costly has our second bloodless revolution been? In our first revolution, in which we turned away from the counting numbers towards the integers, we sacrificed the leader but we still kept the concept of a parade (without beginning or end). Now, in turning away from the integers towards the fractions, we have sacrificed the last remaining characteristic of the counting numbers, namely, the concept of the follower. If you mention any two fractions, say $\frac{1}{2}$ and $\frac{1}{3}$, I know which of these follows the other; $\frac{1}{2}$ follows $\frac{1}{3}$ because 1×3 follows 2×1. But I can average the two fractions; *i.e.,* I can take their sum $\frac{5}{6}$ and multiply it by $\frac{1}{2}$ obtaining $\frac{5}{12}$. This average follows $\frac{1}{3}$ and is in turn followed by $\frac{1}{2}$. Thus between any two fractions there lies a third, namely their average; no fraction has an immediate follower.

Once the similarity between fractions and integers is thoroughly understood, a natural question arises. What corresponds in the theory of fractions to the negative of a number in the theory of integers? Adopting again our visualization by means of coins, we may say that the numerator of our fraction is stamped on the upper, and the denominator on the lower, side of the coin. Then all coins save one may be turned over; the one which is nailed down being that one which has zero on its upper side. We obtain, in this way, from every fraction save zero a new fraction which we term its inverse or reciprocal. For instance, the inverse of $\frac{3}{4}$ is $\frac{4}{3}$. Just as there is one, and only one, integer (namely, zero) which is the same as its negative, there is one, and only one fraction (namely, 1) which is the same as its inverse. Just as the sum of an integer and its negative is zero, so also the product of a fraction by its inverse is 1. This deep-lying parallelism finds its true expression in the theory of logarithms, about which we shall say nothing further here.

In this outline of the things which I think are essential in the teaching of elementary arithmetic and algebra, I have only mentioned the two basic operations of addition and multiplication. Why have I ignored subtraction and division? Well, if I had my way, I should outlaw these terms just as we have had to outlaw fractions whose denominator is zero. It is a delusion to think that we can subtract in the republic of counting numbers, or divide in the republic of integers. For this delusion creates class distinctions which are opposed to the very idea of a republic; we can subtract 3 from 8, but we can not subtract 8 from 3; and we can divide 4 by 2 but we can not divide 3 by 2. Once this is clearly realized, the very reason for the concepts of subtraction and division evaporate. If we wish to subtract we change the sign and *add;* in other words, we never subtract, we always add. If we wish to divide (a thing which can only be done in a satisfactory manner under the constitution of the republic of fractions) we invert and multiply; in other words, we never divide, we always multiply. I feel that this is more than an aesthetic question about which argument is profitless. When I ask my students to solve the equation $2x=8$ I never have, in a quarter century's experience in teaching, received a wrong answer. But when I ask how the correct answer $x=4$ is obtained, I am told, without a single variation in the answer, that the 4 is obtained by dividing 8 by 2. When I ask what this means, I usually fail to receive anything more illuminating than a look of indignation. I hope sometimes to be told that the four was obtained by multiplying both sides of the equation $2x=8$ by $\frac{1}{2}$; and my conviction that a better era of teaching has arrived will be complete if my student adds that it is a very comforting fact about multiplication that $\frac{1}{2}\times 2x=\left(\frac{1}{2}\times 2\right) x=1\times x=x$. He will thus show me that he has not only heard about, but has to some extent realized the importance of, the associative law of multiplication. When I ask this future student of mine what $\frac{1}{2}$ is, he will not say one divided by 2; he will say that it is the inverse of $\frac{2}{1}$ or 2. And in explaining what he means by $\frac{3}{2}$ he will be too well taught to imagine that by some magic he has learnt how to divide 3 by 2. He will say simply that $\frac{3}{2}$ is the product of $3=\frac{3}{1}$ by the inverse $\frac{1}{2}$ of $2=\frac{2}{1}$.

So far we have discussed only those matters which belong to the elementary schools and have left little time for the topics taught in the high schools and colleges; for this we make no apology for "as the twig is bent so grows the tree." In the high school the student should learn that further peaceful revolutions are necessary before he understands what is meant by an irrational number such as $\sqrt{2}$ and by complex numbers. For these latter numbers (which are so important in such practical applications of mathematics as radio transmission) the revolution is exactly of the same type as those which introduced the integers and the fractions. Let us devoutly hope that no longer will a teacher, after telling a student that the square of a number is always positive, insist that he manipulate the square root of minus one. It is little comfort to a properly rebellious student to say that he may salve his conscience by terming this mysterious symbol an imaginary. A good teacher will tell his students, when they are studying algebra, that there is not merely one algebra but that many algebras exist. It is a matter of national pride that America is now the foremost contributor to the theory of Modern Algebra, that wonderful creation of the human mind which has proved so fundamental for the most recent advances in physics.

In closing this address, I think that I should try to forestall the following criticism: Here is a college or university professor engaged in the age-old pastime of passing on the blame. All he can find to do is to criticize the teaching in the more elementary schools. I can assure you that I am not very happy about the way mathematics is taught in our colleges. In a long experience, I have had many graduate students, all of whom had studied calculus. Believe it or not, most of them are surprised when I tell them that in studying calculus they have been learning how to calculate; and that the adjective differential in the term differential calculus has something to do with the word difference. To drop to the freshman level, I find very few amongst the multitudes studying trigonometry, *i.e.,* the science of measurement of angles, who know what an angle is. The few who do know in a vague way that it is the length of a circular arc have never been told that the length of a circular arc is a difficult concept whose mastery marks an important stage in the development of an understanding of mathematics.

The net result is that a large part of our teaching of mathematics in college has degenerated into a mechanical formalism under which the best students can differentiate and integrate the most elaborate expressions, frequently without a clear understanding as to what is going on. What does it mean to say that a function is differentiable or that it is integrable? When I ask this question, I want a clear-cut answer in simple terms, not the parrot-like repetition of some definition in a text book. There is far too much talk in the teaching of mathematics. If I were an architect designing a mathematics class room, I would have cut in large letters above the blackboard the motto:

Cut out the talk; what have you got?

To those of you who teach mathematics and who say to me that my program is too idealistic and that one simply can not teach mathematics properly, I must point the accusing finger. Study well your responsibility before you poison the wells. Do not, I urge you, be so pessimistic. It is as easy (really much easier) to teach mathematics correctly as to teach it incorrectly; and I can assure you that it is much more fun. As a born Irishman I am entitled to close the book for the day, before the evening is too far advanced, and looking up, say to my good companions: Let's have a little fun. *Beannacht leat;* God be with all here, may He bless the work and the fun.

CULTURAL INTERCHANGE BETWEEN THE SOVIET UNION AND THE UNITED STATES[1]

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THROUGHOUT many generations in which creative intellectual activity was the privilege only of the few in Russia, Russian scientists, authors and composers nevertheless produced works of the highest quality which have greatly enriched the culture of the entire world. Mendelejeff, Metchnikoff, Iwanowski, Winogradsky, Pavlov, among others, may be mentioned as examples of men whose contributions are basic to modern chemistry, agronomy and medicine.

In the Soviet Union education and pure scientific and technological training have become the privilege of the many and have been vigorously fostered by the government. A great development of mathematics and physics, agriculture, geography, geology, the biological and medical sciences and the technologies has resulted and has afforded the foundation without which Russia's magnificent achievements in the present war would never have been possible.

What the Soviet Union has actually achieved in pure science and technology within a single generation may be taken as a measure of how great her contribution to world culture and well-being may become under conditions of peace. Science can discover and develop the necessary means for material comfort and well-being; no well-informed person seriously doubts that, I believe. There are, however, far subtler and more complex problems for the solution of which scientific method and scientifically minded men must give their best efforts. For instance, living which brings satisfaction and creative possibilities to the individual and the group is by no means solely the result of material well-being, but of complex physiological and emotional adjustments producing inner harmony; the conditions of this inner harmony I sincerely believe are ultimately discoverable by scientific method. The complex conditions necessary for economic and social adjustment and well-being, I believe, too, are ultimately discoverable through rational methods, and are in considerable measure capable of achievement, in a world of men of good-will.

Any scientist knows that such complex problems as these require the joint effort of many people, with many divergent backgrounds, working from many different angles of approach. From the point of view of the social or natural scientist, therefore, diversity of experience and of social and economic organization is to be welcomed and valued. We need the common efforts and friendly rivalry of Soviet and American, of French, British and Chinese and every other kind of social and natural scientist if a better and more harmonious world is to evolve.

May I repeat. To the scientist striving for understanding and peaceful evolution, diversity is welcomed. Areas of difference between people of different nations should be precisely and rigorously defined in order that apprehension should be confined within the limits thus prescribed, leaving the whole world of ideas outside these limits as the common heritage of mankind. It is when the fear arises that revolution or conquest may impose an alien order by coercion that suspicion and ill-will appear. Let every citizen of our two great countries resolve that change shall henceforth be by cooperative, peaceful evolution and never by such brutal conquest as our common enemies have attempted.

I am asked to speak concretely about the cooperation that now exists between American and Soviet scientists and the means by which this cooperation could be made more fruitful in the future. Cooperation at present is terribly handicapped by the exigencies of

[1] Address before the dinner of the American Russian Institute, New York, October 20, 1944.