## ON THE CELLULAR DIVISION OF SPACE WITH MINIMUM AREA

The problem of the division of space into congruent cells of minimum area has been studied previously. By means of the Calculus of Variations, one may demonstrate easily that such cells must have dihedral angles of 120°. The rhombic do-decahedron is a figure which possesses the characteristic of filling space and having all its dihedral angles of 120°, and on that account was considered for a long time as the

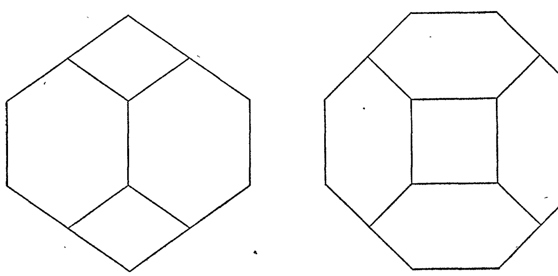

FIG. 1.

solution of the problem. One of the facts that raised a doubt as to whether it might be is that in nature, in all the cases of division of space in which the surface tends to diminish in area, there have been further hexagonal faces. The problem was resolved, however, in decisive form by Lord Kelvin.[1] Basing his argu-

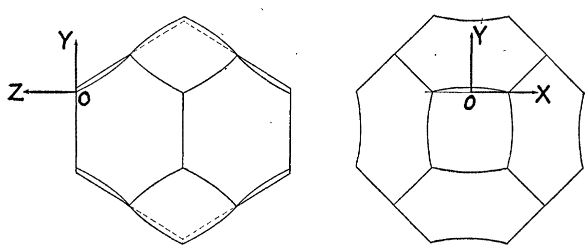


FIG. 2.

ment on the physical experiments of Plateau on soap films, he deduced that the figure must be a curved 14-hedron similar to the planar, so-called orthic, 14-hedron. The planar 14-hedron can fill space, and for the same volume has less area than the rhombic dodecahedron, but its angles are not 120°.[2] By means of the Calculus of Variations Kelvin determined the form of this curved polyhedron of minimal area.

This problem interests greatly the biologists by its relation to the shape of cells in animal and plant tissues, in particular bean cells. Nevertheless, some biologists appear to doubt that the minimal 14-hedron of Kelvin has less area than the planar 14-hedron. Dr. Frederic T. Lewis, of the Harvard Medical School, who has studied the problem deeply and has been much interested in the solution, told me of this uncertainty as to the true solution. In order to dissipate the doubt of the biologists, I have not calculated the area of the 14-hedron of Kelvin, which would be quite a laborious task, but have constructed another curved 14-hedron with the same volume as the planar 14-hedron, and which has less area.

The form of construction, starting from the planar 14-hedron, is the following: in the plane of the square face with coordinates such as are indicated in Fig. 2, one takes a curve $y = k\,f(x)$ such that $f(-x) = f(x)$, $f\left(\frac{l}{2}\right) = 0$. One constructs a conical surface with vertex in the center of the adjacent hectagon with the curve as directrix. If one repeats this operation with all the edges of all the squares, the figure that one obtains has the same volume as the planar 14-hedron and can fill all space, as one sees easily from the symmetry of the figure (having all the symmetries of the planar 14-hedron). Its area will be a function of the parameter $k$, and to determine its minimum value is a very simple problem. For the demonstration, I have taken $l = 2$, and $f(x) = 1 - x^2$. The area is given by the formula

$$A(k) = 24 + 32\,k + 24 \int_{-1}^{+1} \sqrt{3 - k\left[2(1+x^2) - k(1+10\,x^2+x^4)\right]}\,dx$$

Since

$$\sqrt{a^2-b} = a - \frac{b}{2a} - \frac{b^2}{2a\left(a+\sqrt{a^2-b}\right)^2};$$

and neglecting $b$ in the denominator of the final term, one finds that the value of $k$ which gives the minimum area is $k = 0.044$. Observing that

$$(2-k)k \leqq k[2(1+x^2) - k(1+10\,x^2+x^4)] \leqq (4-12\,k)k$$

for $-1 \leqq x \leqq 1$, one finds the area of the figure is:

$$A = 24 + 48\,\sqrt{3} - 0.1106 \pm 0.0001$$

which is 0.103 per cent. less than the area of the primitive polyhedron.

Dr. George D. Birkhoff has inquired concerning the interesting case $f(x) = 1 - |x|$ in which one has a polyhedron of 54 sides (but only 14 faces) which presents the required space-filling characteristics and for $k = 0.059$ is 0.086 per cent. less area than the original 14-hedron.

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[1] See Sir W. Thomson, "On the Division of Space with Minimum Partitional Area." *Philosophical Magazine and Journal of Science*, vol. XXIV(5), 1887.

[2] For exact dates and careful description, see D'Arcy W. Thompson book "On Growth and Form," Cambridge (Eng.) at the University Press, 1942.