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# A MATHEMATICAL THEORY OF EQUILIBRIUM WITH APPLICATIONS TO MINIMAL SURFACE THEORY[1]

By Professor MARSTON MORSE

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THE theory of equilibrium points or critical points of functions appears in fragmentary form in the work of Poincaré, Maxwell and Kronecker. Birkhoff introduced the minimax principle and applied it in dynamics. A systematic study of the critical points of functions of $n$-variables was begun by the author in 1922. A. B. Brown made important contributions. In 1924 the calculus of variations in the large was introduced and developed as an extension of the theory of critical points of $n$-variables. The integrals used were ordinary and were regarded as functions of the curves along which they were evaluated.

[1] Address of the retiring vice-president and chairman of the Section on Mathematics of the American Association for the Advancement of Science, Philadelphia, December 27, 1940.

In 1937 the theory was put on a more general basis with the function defined on an abstract metric space. For details the reader may refer to the author's fascicule on "Functional Topology and Abstract Variational Theory," published by Gauthier-Villars, Paris. In this general theory one is free from any limitation of dimension. It is immaterial whether the independent variable is a point in a Euclidean space, a curve, a surface or a more general configuration. A critical point of a function of $n$-variables is a point at which all the partial derivatives vanish. When the function is an integral in the calculus of variations—for example, the integral of length on a surface or the area of a surface bounded by a curve—a critical point (curve or surface) is one satisfying the Euler-Lagrange equations. The unification which appears in the general theory is made possible by a topological definition of a critical point, a definition which is independent of the particular case at hand.

The years 1929–1936 saw a rapid development of the theory in the large of integrals depending on a curve. There remained the multiple integrals which are so important in mathematical physics. A typical integral is the Dirichlet integral $d(x)$, the double integral over a plane region of the sum of the squares of the partial derivatives of $x(u, v)$. For our purposes we need the Douglas extension of the Dirichlet integral to a surface. $x_i = x_i\ (u, v)\ i = 1, 2, 3$. This integral is denoted by $D(x)$ and defined as one half the sum $d(x_1) + d(x_2) + d(x_3)$. The critical surfaces here are termed *minimal* surfaces. They are surfaces which locally have the shape of a soap film. At all ordinary points $p$ of such a surface the two centers of curvature lie on opposite sides of the tangent plane at $p$ and at the same distance from this plane.

The integral $D(x)$ is as typical an integral as one could hope to find, but the difficulties in treating it were very great. It was necessary to verify the conditions of the general abstract theory in order to apply that theory. This was accomplished independently by Morse and Tompkins and by Shiffman in papers appearing in 1939–40 in the *Annals of Mathematics.* Morse and Tompkins made use of the thesis by Lebesgue and of the work of Douglas and Radó on the minimum problem: Shiffman made use of the papers of Courant.

The case of surfaces bounded by a single closed curve without self-intersection was first treated. The surfaces for which $D(x)$ is evaluated are defined for parameters $(u, v)$ which range over a unit disc $u^2 + v^2 \leqq 1$. Thus they are "topologically of the disc type." When one turns to surfaces bounded by several contours the minimum problem has been treated by Douglas, Courant and Shiffman in a number of papers. The difficult theory of unstable minimal surfaces bounded by several contours has been studied by Morse and Tompkins in a paper which will be published shortly. An abstract of this paper will appear in the December number of the "Proceedings" of the National Academy, where references to other recent papers may be found. Shiffman has also completed an independent paper on the case of several contours, and this paper will presently be published. I understand from a brief verbal account of his work by Shiffman that the treatment by Morse and Tompkins differs radically in the choice of the independent variables.

This difference may be described as follows. The surfaces on which the integral $D(x)$ is evaluated are harmonic surfaces, that is, surfaces $x_i = x_i\ (u, v)$ such that $x_i\ (u, v)$ satisfies the Laplace equation for each $i$. No generality is thereby lost because it is known that minimal surfaces are a sub-class of harmonic surfaces, in fact, harmonic surfaces on which the parameters $(u, v)$ form an isothermal net. Let $(r, \theta)$ be polar coordinates in the $(u, v)$ plane. In the case of a single boundary $g$ the harmonic function $x_i\ (u, v)$ is completely determined by its values $p_i\ (\theta)$ on the circle $r = 1$. But the only surfaces admitted are those bounded by $g$. Thus $x_i = p_i\ (\theta)$ is a representation of $g$. Different representations of $g$ will give different harmonic surfaces. We use only those representations of $g$ which are obtained from a one-to-one representation of $g$ by a continuous monotone transformation of the circle $r = 1$. These representations $(p)$ of $g$ determine the harmonic surface and hence $D(x)$, so that we may write $D(x)$ as a function $A(p)$ of the representation $(p)$. This is the point of view of Douglas in his minimum theory.

When one has several contours, say two contours $g_1$ and $g_2$, the unit circle in the $(u, v)$ plane is naturally replaced by a corresponding number of closed plane curves. Suppose, for example, there are two non-intersecting closed curves bounding a region $S$ in the $(u, v)$ plane. It is known that any such region $S$ can be conformally mapped onto a ring $R$ bounded by two concentric circles, the ratio $\rho$ of the radii of the two circles being determined by $S$. The harmonic surfaces admitted will be defined for $(u, v)$ on $R$. We may suppose that $g_1$ has a representation $x_i = p_i\ (\theta)$ in terms of the angular coordinate $\theta$ of a point on the outer circular boundary of $R$. The curve $g_2$ is similarly represented on the inner boundary of $R$. The integral $D(x)$ taken over $R$ will be a function of the above representation of $g_1$ and $g_2$ and of $\rho$.

But difficulty arises due to the fact that as we vary the representations $(p)$ the functions may converge in the mean to functions which are identically constant (termed degenerate). If one should restrict our representations $(p)$ of a curve $g$ on a circle $C$ to those which make three fixed and distinct points of $g$ correspond to three fixed and distinct points of $C$ this convergence to degenerate functions would not occur. We term such representations *restricted.* Such a restriction of our representations is not possible if there are two or more contours because some of the minimal surfaces sought might thereby be excluded. The difficulty is greatly eased by Morse and Tompkins by a new choice of the independent variables as follows.

By a Möbius transformation of a circle $C$ into itself is meant the transformation $\theta' = t\ (\theta)$ induced on $C$ by a classical Möbius (directly conformal $1-1$) transformation of the corresponding circular disc into itself. A restricted representation $p_i\ (\theta)$ of a boundary curve $g$ on a circle $C$, if preceded by a Möbius transformation $t(\theta)$ of $C$, will yield a general representation $p_i[t(\theta)]$ of $g$. This representation is formally written $pt$ and termed a product representation. Morse and Tompkins use these product representations whenever there is more than one contour. One advantage lies in the fact that the restricted representations converge for bounded $D$ only to restricted representations, while the Möbius transformations converge in a simple manner to Möbius transformations or degenerate transformations regardless of whether $D$ is bounded or not. Another advantage is that $D$, regarded as a function of these restricted and Möbius transformations and of the ratio $\rho$ depends continuously on the Möbius transformations and $\rho$.

With these new variables the definition of the integral $D$ can be extended by a unique limiting process to the cases where $\rho = 0$ or where one or more of the Möbius transformations are degenerate. The resulting function of these new variables is shown to satisfy the conditions of the general theory, which may then be applied.

We shall give a typical theorem, using conditions somewhat less general than those which will appear in our paper, in particular not introducing the "chord arc condition."

THEOREM. *Let $g_1$ and $g_2$ be two simple closed curves with continuously turning tangents. Suppose that $g_1$ and $g_2$ are separated by a plane and have convex projections on suitably chosen planes. If $g_1$ and $g_2$ bound a ring type minimal surface of minimum type they also bound a ring minimal surface of non-minimum or unstable type.*

The conditions of the second sentence of the theorem can be considerably lightened. The theorem may be illustrated by a classical example in which the boundaries are two circles with centers on a common axis and planes orthogonal to this axis. If the planes of these circles are near enough together the circles bound two minimal surfaces of revolution generated by catenaries. One of these surfaces is of minimum type and the other of non-minimum type.

We have not gone into great technical detail in our paper. We have rather presented general fundamental ideas by way of typical examples.

These ideas and examples we believe may be a prelude to far-reaching applications both in mathematics and in mathematical physics.

# TECHNICAL PROGRESS IN AVIATION[1]

By Dr. J. C. HUNSAKER

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THE increasing importance of the airplane in our normal social and economic life is just now overshadowed by its dominant part in our national security. When recent advances in the aeronautical sciences are being tested on a grand scale in the proving ground of war, we naturally lose interest in the possibilities of those same advances in applied science as they might assist our peace-time communications. If we hope to live in a kind of world in which individual living is worth while, our thought must be focused on the airplane as an instrument of air power; destructive or protective.

That important technical progress has been made in the development of the airplane is obvious from the reports of dive bombings, night bombings and aerial torpedo attacks on the one hand, and from reports of effective defensive action by fast fighting planes on the other hand. Guns, armor, radio, fuels, meteorology, metallurgy and photography are also contributing through technical advances to the effectiveness of the airplane in war. Attack and defense for colossal stakes stimulate development in every branch of applied science.

Technical progress in the development of the airplane as a vehicle of the air has been especially marked in the past ten years. Some steps in this progress have been abrupt, because they were the consequence of inventions. True inventions are unpredictable, but it is our experience that when new knowledge obtained by research and experiment becomes generally known, the invention necessary for its practical application soon follows. Whether the invention can then be applied by designers to realize a technical improvement in the airplane depends on whether the state of the art is ripe for it.

For example, the aerodynamic advantages of an unbraced monoplane wing were demonstrated by Fokker in the days of braced biplanes, but the safe construction of such wings had to wait for the availability of light aluminum alloys. Likewise the advantage of retracting the landing gear and wheels was recognized at an early date and mechanisms for retraction were invented, but no designer would bother with them until speeds were high enough to make it worth while to accept the added cost and complication. The designer of transport planes, moreover, needed a thick cantilever wing to afford space enough to house the wheels in the retracted position. Consequently, there was a lag of some ten years in the general adoption of this improvement.

Though some technical improvements are thus proposed before the art has sufficiently advanced to permit their use, other improvements come about as a

[1] Address of the retiring vice-president and chairman of the Section on Engineering of the American Association for the Advancement of Science, Philadelphia, December 31, 1940.