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RECENT GENERAL TRENDS IN MATHEMATICS[1]

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EXCEPTING some comparatively isolated researches in elementary geometry and algebra, mathematical research in the first decade of the twentieth century was largely concerned with subjects whose roots were in the theory of functions of real or complex variables and in a large number of special functional transformations of such functions. It is true that in both America and Europe, the mathematical world was then dimly aware that there was such a thing as a theory of functions of abstract (or general) variables, but outside of a few distinguished workers, notably E. H. Moore in America and M. Fréchet in Europe, no one seemed to have done anything about it. This trend towards general function theories made itself felt during the next two succeeding decades in certain branches of functional analysis, topology and algebra. It was not, however, until the last ten years that general analysis and general algebra permeated, or at least influenced, practically every nook and corner of mathematics and symbolic logic. The American and Polish schools of abstract thought have played a leading role in this development. It is gratifying to see that a large number of young men in American centers of learning are making important contributions to general analysis and general algebra. In the interest of clarity it should be remarked in this connection that excellent progress has been made during the last decade in the theory of functions of real and complex variables and their application to various topics in functional equations, to the calculus of variations and to classical differential geometry. The point we wish to emphasize here, however, is that much of this progress was directly or indirectly inspired by ideas current in general analysis.

One of the reasons why general analysis and general algebra are so far-reaching and very concrete indeed when understood is that one can, by special interpretation of a few general variables and operations, obtain old as well as numerous new results by methods that brush aside the unessential and historically acci-

[1] An address delivered by invitation at the University of Illinois, April 30, 1940.

dental. For example, theorems on functional equations $y=f(x)$ in abstract linear metric spaces can by a few master strokes be made to furnish results in such diversified fields as simultaneous equations in real, complex and hypercomplex variables, ordinary differential equations, partial differential equations and integral equations.

One of the most dramatic examples of the influence of general function theory on a branch of mathematics is furnished by coordinate geometry and differential geometry in particular. The words and phrases coordinates, coordinate systems and transformations of coordinates are on the lips of every student that takes the most elementary college mathematics courses. I say "on the lips of every student" but not "understood by every student" for the simple reason that these notions have only recently emerged from prison walls of philosophical and physical fog into the clearer realm of mathematical thought. It must be recalled that previously the same prison walls had kept fettered the whole subject of geometry until the dawn of our present century, when it became increasingly clear that geometries as mathematical subjects must be developed from a set of postulates concerning classes of more or less undefined (abstract or unspecified) objects. The points of a geometry are undefined, but so are certain distinguished coordinate systems. For example, in plane Euclidean geometry of freshman and sophomore fame, a rectangular cartesian coordinate system is a name given to a postulated single valued function $x(P)$ defined over the whole Euclidean plane and having values in the class of matrices $x=(x_1, x_2)$ with real number elements x_1 and x_2, called rectangular cartesian coordinates of the point P. The function $x(P)$ of the abstract variable P is one of a class of such functions that satisfies a few well-known postulates—and outside of the properties demanded by these postulates, the function $x(P)$ is unspecified. In other words, a rectangular cartesian coordinate system is a mathematical "camera" that photographs the whole Euclidean plane and produces a photograph containing the whole world of matrices $x=(x_1, x_2)$. The nature of this mathematical camera is left unspecified—it is merely required to do the job in a manner specified by the postulates. Other coordinate systems, such as polar coordinate systems, are defined by functions of type $f(x(P))$, where $f(x)$ has arguments and values in subclasses of the class of matrices $x=(x_1, x_2)$. We shall not pursue the subject further—our main object here was merely to indicate how fundamental general function theory is even for the elements of plane Euclidean geometry.

We have already alluded to the influence of general function theory on differential geometry. General function theory enters the subject of finite dimensional differential geometry via the theory of topological spaces and especially through Hausdorff topological spaces. A Hausdorff topological space H consists of a class of undefined elements, called points, and supports a topology determined by a class of undefined subsets of H, called neighborhoods, associated with each point of H such that the following four postulates are satisfied: (1) a neighborhood contains the point of which it is a neighborhood; (2) each point contained in an arbitrarily chosen neighborhood can be surrounded by a neighborhood that is contained in the chosen neighborhood; (3) the intersection of two neighborhoods of a point contains a neighborhood of the same point; (4) each of two distinct points can be surrounded by a neighborhood having no points in common with the other neighborhood. A simple example of a Hausdorff topological space is furnished by the Euclidean plane when neighborhoods of a point P are taken to be the interiors of concentric circles with center at the point P. A coordinate system in a finite dimensional differential geometry is a function $x(P)$, usually required to be a homeomorphism, defined on a subset of a Hausdorff topological space and having values in an n-dimensional arithmetic space, called coordinate space, with points $x=(x^1, x^2, \dots, x^n)$. In other words $x(P)$ maps a portion of the Hausdorff topological space into a set contained in a class of one-rowed matrices with n real number elements. It is convenient for what we shall have to say later, and in conflict with the usual terminology, to *define the coordinate of a point* P in a coordinate system $x(P)$ to be *the corresponding value of the function* $x(P)$. A coordinate of a point in an n-dimensional geometry *is then,* according to our new terminology, *a one-rowed matrix* of n real numbers. The classical differential calculus for real functions of several real variables now enters the stage and is applied to the differentiability of the components of postulated geometric objects (for example, the components $g_{ij}(x^1, \dots, x^n)$ of the fundamental metric tensor field in a Riemannian differential geometry) and to the differentiability of the functions involved in transformations of coordinates.

As is to be expected, general function theory plays an even more extensive role in infinite dimensional functional differential geometries. A *coordinate* of a point of a Hausdorff topological space H is now an element of a function space so that a coordinate system $x(P)$ maps a portion of the geometric space H into a class of functions contained in a function space. The differential calculus that is employed in these studies is the classical differential calculus of *functionals.* By a functional we mean here a function whose arguments and values lie in function spaces of functions of a finite number of real variables.

Above all, the influence of general function theory in differential geometry has been felt most strongly in general (abstract) differential geometry, a branch of mathematics which is still in its infancy but which we are convinced is destined to become one of the great branches of mathematics, comparable to the present status of general (abstract) algebra and general analysis. By a *general differential geometry* is meant a differential geometry with an abstract coordinate space. These geometries are *dimensionless* in the sense that the dimensionality of the coordinate space is left unspecified. This makes possible the inclusion of large portions of finite dimensional as well as of infinite dimensional differential geometries as instances of general differential geometries. The abstract coordinate space must be capable, however, of supporting an appropriate abstract differential calculus. It is for this reason that, in the present state of knowledge, the permissible coordinate spaces are topological abelian groups: linear topological spaces and linear metric spaces (Banach spaces). If a general differential geometry has a Banach space as a coordinate space, a coordinate system $x(P)$ maps a portion of the geometric space, a Hausdorff topological space, into a portion of the Banach space, while a transformation of coordinates $\overline{x}=f(x)$ maps a portion of the Banach space into a portion of the Banach space. The function $f(x)$ must in addition have suitable differentiability properties in accordance with the notions of the Fréchet differential calculus of functions with arguments and values in Banach spaces. This makes possible the study of a general tensor analysis and its application to the subject on hand.

One of the interesting features of the work in general differential geometry is that it suggests new advances in the abstract differential calculus, and in abstract differential equations in particular, at almost every turn of the geometrical theory. To a certain extent this is analogous to the situation that must have existed at the time of Gauss and Riemann in finite dimensional differential geometry and the classical differential calculus. Another interesting feature is the large number (roughly 100) of postulates that is required for any one brand of general differential geometry—say, a general Riemannian differential geometry. This is to be contrasted with the handful of postulates required in abstract algebra, say, in group theory.

Mathematical research has profited greatly in the past by the union of two great streams of mathematical thought to form a new vigorous branch of mathematics. Abstract algebra, especially abstract groups, and abstract topology furnish excellent illustrations. What would be more natural than to take the postulates for an abstract group, to add to them the postulates for a topological space and then to require that the group operations be continuous with respect to the postulated topology? The hybrid object that is obtained in this way is now called an (abstract) *topological group.* During the last decade, topological algebras and especially topological groups have been studied with much success for their own sake as well as for the sake of their applications to other branches of mathematics. We have already spoken of the place of topological abelian groups in general differential geometry. This modern work on topological groups has done much recently to clarify Lie's theory of finite continuous groups. There is now in the literature the beginnings of a general theory of continuous transformation groups in abstract spaces with an abstract parameter. The real flavor of Lie's theory lies in its three fundamental theorems, in its differential operators and their commutators and in its structural constants. With the aid of the abstract differential calculus, a certain amount of success has been met in keeping the flavor of the Lie theory in the general theory of continuous transformation groups with abstract parameters. Much remains to be done here.

The applications of group theory are varied and many. The group-theoretic foundations of various elementary geometries have been studied for quite a long while. During the last decade or two much progress has also been made in group-theoretic studies in algebraic geometry and differential geometry. Even if the geometric space of a differential geometry does not have a non-trivial group of automorphisms, there are group-theoretic problems of some importance that arise in such geometries. These considerations would necessitate a radical revision of the seventy-year-old Erlanger Programm. The applications of group theory to physics have been numerous during the last ten years, especially in the quantum mechanics of many electron atoms and of polyatomic molecules. Continuous groups have been used very little in engineering. However, during the last decade many advanced branches of mathematics have found important applications in engineering, and one may suspect that continuous groups would be found helpful in attacking some of the more difficult problems of elasticity, hydrodynamics, aeronautics and meteorology.

There is another aspect of group theory, however, which in recent years has become fascinating. I am speaking here of the applications of various branches of mathematics to group theory. Of course, the application of differential equation theory to continuous groups is well known and we have spoken of the application of topology to continuous groups. Not so well known is the study of topological groups with the aid of the results of modern differential geometry. In fact, a finite dimensional coordinate topological group furnishes a most beautiful and important example of a non-Riemannian geometry, and it becomes Riemannian whenever the group is semi-simple. The theory of integral equations and the modern theory of integrals has been found useful in compact groups. There is a branch of abstract algebra called lattice theory by some and structure theory by others. A lattice is really a generalized Boolean algebra or a generalization of the algebra of point sets. The way in which lattice theory enters into various branches of mathematics is briefly as follows. Many fundamental theorems in a mathematical discipline often deal directly with distinguished subsets of elements rather than with the elements themselves. Now these distinguished subsets in various mathematical domains are found to possess common properties. So there is value in studying a discipline in which an undefined element is abstracted from a distinguished subset of elements of a totally different discipline—and that is what lattice theory does. For example, invariant subgroups are such distinguished subsets of elements in group theory. It is problematical, however, whether pure lattice theory would in the future have as much influence on group theory as some other branches of mathematics have already had.

There are very few branches of mathematics that have not been applied to the solution of problems in the pure physical, biological and social sciences, and the day is dawning when the same could be said about the engineering sciences. I mean this statement to be as sweeping as possible; for there are indications in the scientific literature that show that advanced portions of function space theory and non-linear functional equation theory will, with all their topological paraphernalia, become very useful in practical engineering problems.

Physical and astronomical research from ancient times to the present era has exercised a potent influence on mathematical research. This influence has often been in the form of hunches and heuristic guides. An admirable illustration of this is furnished by the choice of appropriate boundary conditions for a partial differential equation, say, for those equations occurring in potential theory. All this has been common knowledge for many years. There is another side to this influence of physics on mathematics that is not so well known and is taking place right before our eyes. I am referring to some of the misconceptions of physicists that have led mathematicians to start interesting mathematical researches. An excellent example is furnished by the so-called ergodic hypothesis of the classical statistical mechanics. Physicists now seem to be unanimous in their opinion that there is no known physical system that operates under the ergodic hypothesis. Even a hasty survey of the recent mathematical literature shows, however, that mathematicians were first led to the study of the so-called ergodic theorems on linear transformations by their own curiosity concerning the ergodic hypothesis. Another illustration is furnished by the numerous abortive attempts of physicists to manufacture unified field theories in general relativity. All these attempts at a unified field theory seemed to be physically vacuous or untenable, and yet some of the most beautiful work by mathematicians in the field of modern differential geometry was directly inspired by such misconceived attempts.

In conclusion, we would like to make a few more remarks concerning this present-day trend towards more abstract and general work in mathematics—a trend that results more and more in the omission of the real number system as a necessary part of many mathematical disciplines, and a trend that points the way to the Olympian heights of the stately mountain peaks that loom in the ever-widening horizon of the mathematical world. Until rather recently, many mathematical concepts such as limit point, open set, closed set, continuity, differentiability—just to mention a few—were thought to be intimately connected with the real number system. We now know that such is not the case and that these concepts have natural domains of existence that are much more general than real number domains. The methods of proof used in real number theory often have to be radically changed so as to conform to the more general patterns. All this is interesting, and frequently great difficulties arise in the proofs of theorems; but the more novel and often highly original aspects of a general theory are met in the situations which have no real number analogues or in those whose real number analogues are trivial—the latter cases are frequently accompanied by a large number of non-equivalent concepts that become equivalent whenever the domains are specialized to number domains. We have already mentioned the novel contributions of the general theories to the real variable domain and we need not dwell on it here, but merely mention it for the sake of completeness.

PRECIOUS AND SEMI-PRECIOUS JEWELS—THEIR CHEMICAL AND BEAUTIFYING QUALITIES[1]

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IN the time before records primitive man searched for the beautiful and rare. When we come to early writings we read descriptions of gems, and we find examples of these in ancient burials. As man became more civilized, this search for jewels went on with increasing eagerness. Accompanying these early discoveries, and perhaps as a mark of their unusualness,

[1] Lecture illustrated by specimens, delivered before Sigma Xi at Purdue University, Indiana University and Union College, and before the Rochester and Cornell sections of the American Chemical Society.