SCIENTIFIC BOOKS

INEQUALITIES

Inequalities. By G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA. Cambridge, England, The University Press; New York, The Macmillan Company, 1934; pp. xii + 314. $4.75.

THIS book is devoted to a systematic and critical study of a number of inequalities which are fundamental in mathematical analysis, and to the presentation of a wealth of others which the authors have encountered in their wide experience, many of which have been subjects of their own investigations. It is unique in its field for many reasons, but especially on account of the great variety of results presented and the thoroughness with which the inequalities have been analyzed and generalized. Mathematical investigators will find it an indispensable source of information.

Most mathematicians regard inequalities as auxiliary in character and would perhaps not think of them as constituting a domain of principal interest apart from applications. In reading the book it is a pleasant surprise, therefore, to find that the theory of inequalities is a fascinating subject in itself, and to see how effectively the theory may be systematized and correlated by skilful analysts. The authors have achieved much in this regard, and the results of their efforts indicate the possibility of still further interesting correlations in the future. Their plan is outlined in excellent fashion, with regard to both content and method, in Chapter I, which concludes with some helpful advice to the reader who may be interested in principal results rather than details.

Chapters II–VI contain a systematic theory of generalized arithmetic and geometric means and the relationships between them. The very important inequalities usually designated by the names of Hölder and Minkowski appear as special cases of these relationships. For a finite number of variables the inequalities are treated in Chapter II, for a denumerable infinity of variables in Chapter V, and for functions and integrals in Chapter VI. Chapter III is one of the most interesting in the book. It contains a theory of still further generalized means in which the special function $\phi(x) = x^r$ appearing in the original definition of the authors is replaced by a strictly monotonic function $\phi(x)$. Chapter IX is auxiliary in character, devoted to the explanations of various devices from the calculus useful in deducing inequalities.

In words of the authors "the rest of the book (Chs. VII–X) is written in a different spirit and must be judged by different standards. These chapters contain a series of essays on subjects suggested by the more systematic investigations which precede. In them there is very little attempt at system or completeness. They are intended as an introduction to certain fields of modern research, and we have allowed our personal interests to dominate our choice of topics." Thus Chapter VII is devoted to the proofs of a variety of special integral inequalities which are related primarily by the interesting fact that they can all be established by means of the theory of the calculus of variations.

The material in Chapter VIII has to do with multilinear forms in n sets, each containing a denumerable infinity of variables. For convenience here we may agree that such a set of variables x_i defines a point in a Hölder space if a sum $(\Sigma|x_i|^p)^{1/p}$ with $p > 0$ is finite. The chapter begins with a very general theorem specifying an upper bound for a multilinear form whose variables define points in Hölder spaces related to each other by suitable conditions on the exponents p. The theorem has numerous interesting applications. In the latter part of the chapter bilinear forms with $n = 2$ are more intensively studied. Properties of bounded bilinear forms are deduced; two special bilinear forms of Hilbert are discussed; and a "convexity theorem" of M. Riesz for bilinear forms is developed and applied. Chapter IX is devoted to an important theorem of Hilbert giving an upper bound of the special bilinear form $\Sigma x_i y_k/(i+k)$, with analogues for integrals and with numerous modifications and extensions.

Chapter X contains theorems concerning rearrangements of two or more sets of non-negative numbers, and corresponding theorems concerning rearrangements of functions. A fundamental theorem for two sets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ is that the sum $\Sigma a_i b_i$ is greatest when the notations for the sets are so chosen that the elements of both sets are increasing (or decreasing) in magnitude, and least when their elements vary monotonically in opposite senses. There are similar theorems involving an arbitrary finite number of sets. Departing for a moment from the language of the authors we may define a rearrangement of a function $\phi(x)$ measurable on $0 \leqq x \leqq 1$ as a second function $\overline{\varphi}(x)$ such that for every pair of values y_1 and y_2 the measure of the set of points x at which $y_1 \leqq \overline{\varphi}(x) < y_2$ is the same as that of the corresponding set for $\phi(x)$. A non-negative function $\phi(x)$ integrable on $0 \leqq x \leqq 1$ has a decreasing rearrangement $\overline{\varphi}(x)$ uniquely defined almost everywhere. The theorems concerning rearrangements of finite sets have analogues for functions $\phi(x)$ when sums are replaced by integrals and monotonic rearrangements of sets by decreasing rearrangements $\bar{\varphi}(x)$.

In this connection a paper by Haskins[1] should be mentioned, which seems to have escaped the attention of the authors. Haskins defined (p. 184) the "momental constants" of a bounded measurable function $f(x)$ on an interval $a \leqq x \leqq b$, which except for a constant factor are somewhat specialized cases of the means $\mathfrak{M}_r(f)$ of Hardy, Littlewood and Pólya. He showed (p. 185) that the values of these constants are characteristic of the class of rearrangements of a function, as defined in the last paragraph above, and describes (p. 194) the increasing rearrangement of a function as typical of the class. Furthermore he proves (p. 189) that the means $\mathfrak{M}_r(f)$ have the effective upper and lower bounds of the function $f(x)$ as limits when r approaches $+\infty$ and $-\infty$, respectively. These results are very closely related to some of those given in the book here reviewed. I understand from Professor Haskins that the paper by Schlömilch, referred to in the book, was not available to him in the war-time year, 1916, when his paper was written. Schlömilch's paper deals with similar conceptions for continuous functions and Riemann integration.

No description of the book here reviewed would be complete without mention of the very valuable lists of theorems and examples at the ends of the chapters. If proofs were given for all these results the book would be expanded beyond reason, but in most cases the necessary arguments are clearly indicated or references are cited. This is only one of many features which insure the great value of the book as a contribution to our modern mathematical literature.

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BIOLOGY FOR EVERYMAN

Biology for Everyman. By SIR J. ARTHUR THOMSON. Two volumes; pp. 1561. New York, E. P. Dutton and Company. 1935.

AMONG the biologists living during the last fifty years, perhaps no one has had such wide and diversified interests as the late J. Arthur Thomson. It must be at least forty years ago that a reviewer, contemplating one of his comprehensive works, expressed doubt concerning the possibility of covering so wide a field. He said that he was quite willing to concede that professors knew more about these matters than any one else, and that of all professors, Scottish professors knew most, and yet, after all, what were the limitations of the human mind? At a later date, it was Thomson himself who, in his charming little book on Herbert Spencer, commented on a result of that philosopher's universality; "we can hardly picture the man who has not some crow to pick with Spencer." So it must be, yet with our scientific babel of tongues, it is a saving grace that there are some, if only a few, who can approximate to a universal language and give us an understanding of the whole drama of life, rather than isolated fragments. In attempting to do this, there are two possible methods. One is to condense and simplify, describing vital phenomena in general terms, but not discussing details. Huxley knew how to do this to perfection. But this synthesis, to be rightly appreciated, must rest on a background of knowledge previously acquired. The other method, followed in the book now reviewed, is to describe details in such a manner as to give a vivid impression of living things in all their diversity, while at the same time constantly recurring to the underlying philosophy which relates them to a whole. The reader is stimulated and delighted to discover how much of interest is going on in this world of nature, indeed, in his own immediate vicinity; so much to observe and enjoy which he has not hitherto noticed. Yet as the Reverend Wm. Kirby, famous pioneer entomologist, said over a hundred years ago, all these things can be seen to illustrate the wisdom and goodness of God. We probably do not express ourselves in theological terms, but it comes to much the same thing if we say that we perceive the harmony and unity of nature, the marvelous creative power which we describe as evolution. So we are alternately, or almost simultaneously, analytic and synthetic, guided by the feeling which Tennyson tried to express in his poem on the "Flower in the Crannied Wall." Sir Arthur Thomson knew well how to set these matters forth in interesting languages for the most part intelligible to any educated person. His book is extremely "readable." But neither Thomson nor any one else can simplify biology in such a way as to excuse the reader from any intellectual effort. In truth, we are dealing with the most complex and marvelous phenomena in the universe and those who have grown old in their investigation still feel like beginners. It is this eternal freshness of biology that constitutes one of its principal charms, for those who care to think.

It is encouraging to note that throughout this country there is an increasing interest in biological subjects, an impetus which, when given sufficient opportunity for development, may carry us far. Thomson, in his concluding chapter, sums up the reasons for being interested in biology, as follows:

(1) Biology can spread our table and increase the amenities of life, ameliorating the struggle for existence.

[1] On the measurable bounds and the distribution of functional values of summable functions, *Transactions* of the American Mathematical Society, vol. 17 (1916), pp. 181–194. See also Jackson, *ibid.*, pp. 178–180; Van Vleck, *ibid.*, vol. 18 (1917), pp. 326–330.