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SPINORS[1]

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THE theory of spinors had its origin in the search for a suitable mathematical tool to use in the extension of the quantum theory to the field of relativity. The quantum mechanics in the form that was given to it by Schroedinger describes the motion of a particle by means of the concept of a wave. It is not, as people used to say, that a physicist thinks of an electron as a particle on Mondays, Wednesdays and Fridays, and as a wave on Tuesdays, Thursdays and Saturdays, and on Sundays prays for a Messiah who will lead him back to the belief which he held on Mondays. The actual situation is quite different from that. He works with a mathematical theory which he visualizes for some purposes by means of the classical conception of a particle and for other purposes by means of the imagery of the wave theory. The wave that he works with is just a function which satisfies a partial differential equation of a certain type. The physicist believes that by applying a certain integration process to the solution of this partial differential equation he is able to express the probability that the particle which he thinks of shall be in a certain preassigned position with a certain preassigned velocity.

The whole thing is an attempt to find mathematical formulas and language for the discussion of phenomena which did not make sense in terms of the language and formulas which the physicist had been using before. Some people actually go so far as to say that we shall have to make real changes in our habits of thought and use of language. But I am referring to these deep and difficult questions only incidentally. I am concerned with something much more superficial.

The spinor theory grew out of the attempt to reconcile the wave mechanics with the relativity theory. The wave mechanics was at first developed so as to fit into the framework of the classical dynamics. On the other hand, the theory of relativity has taken

[1] Fourth of the Joseph Henry Lectures of the Philosophical Society of Washington, presented March 31, 1934, in honor of the first president of the Philosophical Society. This paper was prepared from stenographic notes taken at the time of the lecture.

such a firm hold of all branches of physics that every one is convinced that a really sound theory must take it into account. Therefore, the problem was to find a relativistic formulation of the quantum theory.

Also there was experimental evidence indicating that from the particle point of view an electron should not be considered just as a mathematical point but rather as a thing which is capable of a rotary motion or spin at the same time that it has a motion of translation. The problem of bringing this concept into the theory of the electron turned out to be closely related to the problem of giving a relativistic formulation to the differential equations of the electron.

What looks like a very good solution of the problem was developed by Dirac, building upon previous work of Pauli and others. Dirac modified the Schroedinger differential equation not only by changing its form but by replacing it by a system of four equations with four unknown wave functions, ψ^1, ψ^2, ψ^3, ψ^4. These four functions were related among themselves in what seemed to be a very intricate manner, but they were evidently the components of a physical quantity of some sort.

When I speak of a physical quantity I am thinking of something which has components analogous to the components of a vector. When you take the three rectangular components of a velocity you recognize that you are taking components of something which has a physical existence. In the same way, the quantities which appeared in the Dirac equation were evidently components of some sort of a physical object. But they behaved quite differently from the components of any previously known physical quantity and thus provided a puzzle for the mathematical physicist.

The problem was clearly formulated by the late Professor Ehrenfest. He said, in effect: We are familiar with such things as vectors which are the tools of classical physics. Since the advent of the relativity theory we have got acquainted with the theory of tensors and have been led to believe that any physical phenomena could be described by means of tensors. Now comes a new kind of a physical quantity which is not a tensor and yet has to be taken into account. It has something to do with a spinning electron. Let us call it a *spinor*. Then he called on the mathematicians to provide a theory of spinors, if possible, analogous to the theory of tensors.

The elements of such a theory were in fact already available in Dirac's own work and in the previous work of Darwin and Pauli. The Dirac equation had also been adequately discussed from the point of view of the underlying group theory by Weyl in his book on group theory and quantum mechanics, so that implicitly a good deal of the requisite theory was in existence. Nevertheless, so long as it was possible for a mathematical physicist of the order of magnitude of Ehrenfest to regard it as obscure there remained something of a problem.

Promptly in response to Ehrenfest's challenge, a formal theory of spinors was produced by van der Waerden. This was a theory of two-component spinors which was adequate to the Dirac equation in its original form. But here one has to say, as in so many other cases, that a fully satisfactory account of the subject was possible only after the original theory had been highly generalized. It was in fact so in this case. The original Dirac equation was relevant to the special relativity. The extension to general relativity was indicated first by Weyl and Fock and the system of mathematical equations thus determined has been studied by Schouten, Schroedinger, Einstein and Mayer, and other mathematical physicists. From these studies there has now emerged a clear conception of a class of physical objects which we call spinors and which can be precisely defined.

I shall try to state this definition. In doing so I propose to repeat a number of well-known elementary ideas leading up to the one step which introduces the definition. After this is done the whole matter may seem rather trivial, but it is nevertheless true that after this foundation is laid the working out of the theory becomes a matter of technical detail.

We start with elementary geometry. How are the points in a room to be described? The first step in such a description is to give names to the points so as to distinguish them, and we agree to use numbers as names. A point will have a first name x, a second name y and a third name z. The way which we all know for assigning these names is to let x be the distance of the point from the floor, y the distance from the wall at the front of the room, and z the distance from one side wall. That way of assigning the names is of course completely arbitrary. It could be done in some perfectly bizarre way so long as you satisfied the condition of giving different names to different points. A system of naming the points is what we call a coordinate system.

I have mentioned a particular way of naming the points only to emphasize the fact that the particular system we use for assigning these names is of no importance. We can, in fact, when we have one way of naming them, get any number of other ways by the following device. Suppose you write down these three equations:

$$\begin{aligned} X &= x + y \\ Y &= x - y \\ Z &= z^2. \end{aligned} \tag{1}$$

If we substitute in these equations the numbers x, y

and z appearing in the name of this point, we get three other numbers (X, Y, Z) which gives us a new name for the point, and so we have a new coordinate system. These equations can be solved to give

$$\begin{aligned} x &= \tfrac{1}{2}(X+Y) \\ y &= \tfrac{1}{2}(X-Y) \\ z &= \sqrt{Z}. \end{aligned} \tag{2}$$

If we apply these equations to any point in the room and know the names X, Y, Z, we are able to get back the names x, y, z. We have a dictionary which translates one system of nomenclature for our points into another system of nomenclature. This dictionary is what we call a transformation of coordinates.

In general, a transformation of coordinates

$$(x, y, z) \rightarrow (X, Y, Z)$$

is defined if we replace the right-hand members of equations (1) by quite arbitrary functions subject only to the condition that (1) should be capable of being solved so as to obtain the inverse transformation

$$(X, Y, Z) \rightarrow (x, y, z)$$

analogous to (2). In all this we confine attention to points in the room, that is to say, to a limited portion of space. In practice we require that the functions used shall be amenable to the processes of analysis, such as differentiation, etc., corresponding to the technique of the mathematician of the present epoch.

The essential point which I should like to stress in this consideration of coordinate systems and transformations of coordinates is that the coordinate system is something which we ourselves introduce. It is something in addition to the physical state that we are trying to describe and represents our point of view towards the natural phenomena which are under consideration. To be objective, we must somehow or other get away from this thing that we have introduced. In previous generations mathematicians and physicists used to play with the idea of doing without coordinates. The geometry of Euclid is an example in which coordinates were not used and the attempt was made to reason directly with the physical objects we were talking about. Many will recall the time when it was regarded as important to do vector analysis without coordinates. This idea was based on the feeling that by so doing one was dealing with the natural object itself.

An equally good way of being free from the influence of the coordinates introduced is to use all coordinate systems. Using no coordinate system is, so to speak, the dual idea to using all possible coordinate systems. If you arrange your work in such a way that it applies no matter what the coordinate system is, then you have reached the ideal of dealing with the object itself. This point of view has become very common since the discussions which were brought about by the theory of relativity.

In dealing with physical problems much use is made of vectors. A vector is a special case of what we can call a physical object with components. The idea is something like this: Supposing that in a room we have at every point a tendency of a certain sort, no matter what sort, but a tendency in a definite direction with a definite magnitude. That tendency can be defined by associating with each point (x, y, z) three numbers V^1, V^2, V^3.

$$V^1(x, y, z) \quad V^2(x, y, z) \quad V^3(x, y, z).$$

They describe this tendency and they are the components of something that represents a physical state of affairs. If this physical object is a vector, then on making a transformation into a new coordinate system you will get functions

$$\overline{V^1}(\bar{x}, \bar{y}, \bar{z}) \quad \overline{V^2}(\bar{x}, \bar{y}, \bar{z}) \quad \overline{V^3}(\bar{x}, \bar{y}, \bar{z})$$

of the coordinates $\bar{x}, \bar{y}, \bar{z}$. When you measure this physical object in the new coordinate system, you will get the new set of quantities $\overline{V}^1, \overline{V}^2, \overline{V}^3$, and there will be definite formulas which tell you what these components in the new coordinate system are:

$$\overline{V^1} = f^1(V^1, V^2, V^3) \quad \overline{V^2} = f^2(V^1, V^2, V^3) \quad \overline{V^3} = f^3(V^1, V^2, V^3).$$

You will have three formulas of this sort which express the new components as functions of the old ones. There is no need of my mentioning what these formulas are in detail, for in talking about a subject which is full of formulas, we should be hopelessly lost if we got tangled up with particular formulas. The essential point which I want to bring out is that when you change to a new coordinate system you get a new set of components and in every coordinate system there is a set of components for the physical object. If the law which tells you the new components in terms of the old components is of a particularly simple sort, then your physical object is a vector. There are lots of other physical objects. For example, there are physical objects with 9 components that you could call T_{11}, T_{12}, T_{13}, etc., using two indices. Then if you have a certain formula connecting the components in one coordinate system with those in another, the thing you are talking about is a tensor of the second order. The essential things about a tensor are that it is a physical object with components, and that the components are uniquely determined when the coordinate system is given in terms of which the components are described. I am intentionally leaving this statement in a thoroughly abstract form.

When we come to the theory of relativity we must pass from the three-dimensional space of points to the four-dimensional world of events. This is a story which you probably have heard many times. If you want to describe the events which take place in this room, you have to give not merely *x, y* and *z,* which tell you *where,* but also *t,* which tells you *when,* for each event. The essential point is that the events we talk about are things which are capable of being named by means of four names, the four names being numbers. This can be expressed by saying that the events constitute a four-dimensional world or space-time.

Let us transfer what we have just been saying about coordinate systems from the world of points over to the world of events. We make the same remark that we made before. The essential thing about a coordinate system for events is not any particular way of setting up the coordinate system but is the fact that the coordinate system assigns distinct names to different events.

In order to make an objective description of the world of events, we deal with the totality of coordinate systems. We keep free from any particular point of view and so talk about all coordinate systems at once. For this purpose we have a complete theory of transformations of coordinates and a theory of vectors and tensors. A tensor is a physical object such that with every event we are able to associate a set of numbers called its components when we have before us a given coordinate system. If we change to a new coordinate system we get a new set of components of the same physical object.

Thus in the general relativity theory itself we have a set of 16 functions

$$g_{11}(x^1x^2x^3x^4), \quad g_{12}(x^1x^2x^3x^4), \cdots, \quad g_{44}(x^1x^2x^3x^4).$$

These functions of the coordinates are the components of a physical object called the fundamental gravitational tensor. They satisfy a system of partial differential equations, and the theory of these equations is the relativity theory. The general conception is this: We assume that a given body of physical phenomena is representable by a physical object with components of a certain type, and the theory of these phenomena is contained in the set of differential equations which the components satisfy. This, without any formalism, is the basic mathematical idea which appears in the relativity theory.

Continuing in that theory, it turns out that there are certain other kinds of geometrical objects which have to be considered. The ones which appear first are the electromagnetic potentials. Again there are four components

$$\phi_1, \phi_2, \phi_3, \phi_4$$

which are functions of the coordinates. But, as physicists know, when you give the coordinate system the electromagnetic potentials are not fully determined. You can take another function $f(x^1, x^2, x^3, x^4)$ and add the four derivatives of this function to the components, obtaining

$$\phi_1 + \frac{\partial f}{\partial x^1}, \quad \phi_2 + \frac{\partial f}{\partial x^2}, \quad \phi_3 + \frac{\partial f}{\partial x^3}, \quad \phi_4 + \frac{\partial f}{\partial x^4}$$

without changing the physical significance of these potentials.

Let us try to say what is essential in this without using technical language. It ought to be clear even to those who do not know what these differentiation symbols mean. When we specify a definite coordinate system we have not only one set of four functions which appear as the components of our physical object, but we have a whole class of other sets of components. The physical object in question is of an essentially different kind from those which we have previously been talking about. Its components are not fully determined when the coordinate system is given; something in addition has to be specified before the components are known. This additional something which we have to specify we will call a gauge frame.

I might also try to put it in the following way: We previously said that when we introduce a coordinate system we put something into the phenomena of nature, and before we can be talking about nature itself we have to get free of the coordinate system which we put in. When we talk about electromagnetic potentials, we put something else in, namely, the gauge frame, which has to be specified before we can specify the particular set of components which we are talking about.

So our theory has to be such that we make not only transformations of coordinates but transformations of gauge, and we have to formulate our laws of physical phenomena in a manner which is unaltered not only by changes of coordinate system but also by changes of gauge. Physicists have heard a good deal about that under the heading of gauge invariance. The underlying idea is just as before: In trying to describe nature we have introduced not only coordinate systems, but also another extraneous element called the gauge frame. In addition to the theory of coordinate transformations, there is a theory of gauge transformations which has to be recognized in order to free our theory of physical phenomena from this element which we introduced in our view of nature.

The theory of spinors requires another step in this direction. A spinor is a physical object with components. The number of components is a power of four. In a particular case a spinor may have four components $\psi_1, \psi_2, \psi_3, \psi_4$. The components are functions of the coordinates just as the ϕ's and g's were, but when the coordinate system and the gauge frame are given, the components of the spinor are not fully determined. You can take a new set of components $\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3, \overline{\psi}_4$, which will serve equally well as a set of components of this spinor. The new components are given by means of linear formulas in terms of the old components.

$$\overline{\psi}_1 = T^1_1\psi_1 + T^2_1\psi_2 + T^3_1\psi_3 + T^4_1\psi_4;$$

and three other formulas which look like this one. The coefficients T are arbitrary functions. A linear transformation of this sort is called a spin transformation.

When you have given your coordinates and your gauge, there is still something free, which we will call the spin frame, and we are unable to describe our physical object until the spin frame is fixed. We have to state everything that we say about a spinor so that it will be true no matter what spin transformation is applied to the components. A spin transformation is very analogous to a coordinate transformation, but it takes place completely independently of the coordinate transformation.

This is the simplest example of a spinor. There are spinors with 16 components or in general with 4^k components and you will have linear formulas which give you the other possible sets of components in the same coordinate system.

I have not yet mentioned one of the important facts about spinors which give them their significance. Their components are not ordinary numbers. They are complex numbers of the form

$$a + \sqrt{-1}\,b$$

where a and b are real numbers. In this respect they are like other physical objects which appear in quantum theory. There have been cases in physics before where the complex numbers were used as a convenient device but here they come in an essential way.

The additional degrees of complication which appear in the definition of a spinor correspond to the nature of the physical problem which it is designed to meet. Ordinary vectors and tensors would be well enough adapted to tell where an electron is, in what direction it is going, and what its angular momentum is. But the quantum theory states the problem differently. It does not ask directly what these quantities are but rather, what are the probabilities that these quantities shall take on preassigned values. To meet this requirement, it is not the components of the spinors themselves which are interpreted in terms of physical measurements, but certain combinations of these components with their complex conjugates. These combinations of components of spinors are components of ordinary tensors and are interpreted as probabilities that the electron will be in a certain place moving in a certain way.

Let us now repeat the description of a spinor in a few words. A spinor is a physical object which has components which are complex functions of the coordinates. The number of components is a power of four. A set of components is fixed only after (1) the coordinate system, (2) the gauge-frame and (3) the spin frame, are fixed. Whenever (1), (2) or (3) are changed, the components are replaced by linear combinations of themselves according to definite rules.

Suppose that you have spinors with 16 components with two indices, X_{AB}, and suppose that these spinors satisfy the condition that

$$X_{AB} = -X_{BA}, \tag{3}$$

so that they are antisymmetric. Then the mathematicians will recognize that connected with them there is a quadratic expression

$$X_{12}X_{34} + X_{13}X_{42} + X_{14}X_{23} = 0. \tag{4}$$

Those spinors which satisfy this relation have peculiar properties, and it is this quadratic relation which puts the spinors into connection with the fundamental tensor of the relativity theory, because the g's that we have in relativity are also the coefficients of a quadratic expression.

If you are going to describe some particular physical phenomena such as those described by the relativistic theory of the spinning electron, you must pick out one or more particular spinors which embody the physical phenomena in question. It turns out in this special case that you can pick spinors which set up a suitable relationship between the quadratic equation (4) above and the fundamental quadratic form which appears in the relativity theory. The general theory of spinors is the theory of all possible physical quantities of a certain sort. The theory of the electron is the theory of certain particular spinors which describe this electron.

OBITUARY

FRANK LINCOLN STEVENS

WITH the death, on August 16, of Professor F. L. Stevens botanical science has lost one of its most devoted and productive workers; and many younger botanists mourn the loss of his kindly advice and encouragement.

He was born at Syracuse, New York, on April 1, 1871, the only son of H. B. and Helen C. Stevens.