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MATHEMATICS BEFORE THE GREEKS[1]

By Professor R. C. ARCHIBALD

BROWN UNIVERSITY

OUR conceptions of Egyptian and Babylonian mathematics have been notably changed during the past fifteen years so that the third edition of the first volume of Cantor's great history is already quite out of date. Even since last March much of great interest to the student has been published. Hence it seemed both appropriate and timely for me to endeavor to present to you an accurate even though fragmentary survey of present knowledge of the mathematics of the Egyptians and Babylonians, for it is to theirs alone that I shall refer in considering mathematics before the Greeks, that is, before 600 B. C. Indeed, practically all the mathematics which I shall consider was in use before 1600 B. C.

For at least a thousand years prior to 2500 B. C. the non-Semitic Sumerians, who lived just north of the Persian Gulf and south of the Semitic Akkadians, were generally predominant in Babylonia, but were absorbed in a larger political group by about 2000 B. C. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of the rivers regulated by canals, went back to Sumerian times, like also a considerable part of their religion and law, and their system of mathematics, except, possibly, for certain details.

Our knowledge of Babylonian mathematics is derived mainly from tablets in the British Museum, the Prussian State Museum of Berlin, the Ottoman Museum of Constantinople, the University of Strasbourg, the University of Pennsylvania and the Palais du Cinquantenaire of Brussels. Nearly thirty years ago about forty-seven mathematical problems were published in important part IX of "Cuneiform Texts

[1] Retiring address, as vice-president and chairman of Section A—Mathematics, of the American Association for the Advancement of Science, delivered at Des Moines, Iowa, December 30, 1929.

from Babylonian Tablets, etc., in the British Museum" (we shall later refer to this as *CT*.IX), but so far as published material is concerned very little headway has been made in interpreting the problems.

Our knowledge of Egyptian mathematics is gleaned chiefly from two papyri. One of these, called the Rhind papyrus, was written about 1650 B. C., but is a copy of a document which possibly dates back to 1850 B. C., and is preserved in the British Museum, except for some fragments in New York City. Three editions of this have been published: the first by A. Eisenlohr in 1877; the second by T. E. Peet in 1923, and the third by A. B. Chace, L. S. Bull and H. P. Manning in 1927–29. The last-named work contains also my survey of the literature of Egyptian and Babylonian mathematics during the past 225 years.[2] The second mathematical papyrus, called the Golenishchev Papyrus, is in the Museum of Fine Arts in Moscow and was written about 1850 B. C.; but it, too, may be a copy of an older writing of about 2000 B. C. Only five of its twenty-five problems have been published, but through the great courtesy and generosity of a Russian colleague, Professor V. V. Struve, of the Hermitage Museum in Leningrad, I have been furnished with detailed information concerning the reremaining problems of the papyrus.

The earliest dated event in history was the establishment in 4241 B. C. of the Egyptian calendar[3] of twelve months of thirty days each plus five feast days. This action implies a certain use of mathematics at this distant date. Another indication of the primitive development of Egyptian civilization is that writing was in use as early as 3500 B. C., and indeed at this date we find that the Egyptians were already in possession of the decimal system of notation. This is proved by an inscription on a great Hierakonopolis royal mace[4] now to be seen at Oxford. The inscription contains a reference to 120,000 captives and a register of captive animals, 400,000 oxen and 1,422,000 goats. In hieroglyphic writing unity is denoted by a stroke; 10 by an inverted capital U or handle; 100 by a curved rope; 1,000 by a lotus flower, very common in Egyptian fields; 10,000 by an upright bent finger; 100,000 by a tadpole, and 1,000,000 by a god with uplifted hands. Therefore 1,234,567 would be written: a god, then two tadpoles, followed by three bent fingers, followed by four lotus flowers, followed by five ropes, followed by six handles, followed by seven strokes. Such was the writing of large numbers in 3500 B. C. How much earlier was the real beginning of Egyptian mathematics? Did Egyptian and Babylonian mathematics have a common source, or were they independent in origin? It seems as if all that we can at present state with certainty is that by 2000 B. C. we find these two separate highly developed systems, that in the case of one of them a notable stage had been reached 1500 years earlier and that in the case of the other by 2400 B. C., at least, marked development had occurred.

Sumerian mathematics was essentially sexagesimal and while a special symbol for 10 was constantly used it occupied a subordinate position; there were no symbols for 100 or 1,000. One hundred was regarded as 60 + 40, 1,000 as 16.60 + 40, but in this latter case the Sumerian would simply write 16.40. So also 31.6.15 might equal 111,975. In other words, the Sumerians had a positional notation for their numbers. Their symbol for unity was also the symbol for 60, 60^2, for 60^3, etc., or for 60^{-1}, 60^{-2}, etc. Thus 31.6.15 might mean not only 111,975 but also 6,718,-500, or 1866 1/4, or 31 1/10 1/240. When the unit for the first number to the right was known, the whole was determined. The uncertainty in this regard is one of the great difficulties in interpreting Babylonian mathematical texts. Within the period we are considering there was no symbol for zero but a blank space was left,[5] so that 10. .7 might stand for 36,007. In this way another element of uncertainty was introduced in interpreting these ancient writings.

While the largest fractional unit was 60^{-1}, ten of these units gave 1/6 and in ordinary calculations this occupied a central position. In early calculations there were also signs for 1/3(= 2/6); 1/2(= 3/6) and 2/3 (= 4/6), and by about 2200 B. C. there were also signs for 5/6, 1/5, 1/4. So far as our knowledge goes the Babylonians never used such complicated fractions as the Egyptians readily handled.

Babylonian multiplication tables are very numerous and are almost always the products of a certain number successively by 1, 2, 3 . . . 20, then 30, 40 and 50. For example, on tablets of about 1500 B. C. at Brussels[6] are tables of 7, 10, 12½, 16, 24, each mul-

2 In the course of the following foot-notes "Bibliography" will refer to this survey.

3 J. H. Breasted, "Ancient Times," Boston, 1916, p. 45.

4 J. E. Quibell, "Hierakonopolis," pt. 1, London, 1900, plate 26B.

5 The sign replacing the blanks seems to have been first used about 250 B. C.; see the copy of a British Museum tablet in F. X. Kugler, "Die Babylonische Mondrechnung," Freiburg-i-B., 1900, plate 5, l. 26, sign, 3, l. 88, sign 3, l. 89, sign 8, etc.

6 L. Speleers, "Recueil des Inscriptions de l'Asie Antérieure des Musées Royaux du Cinquantenaire à Bruxelles," Brussels, 1925, p. 29, 94–95. In the University of Pennsylvania there are twenty-three tablets containing multiplication tables, in part or complete, and dated for the most part about 1300 B. C.; but a few are dated 2000 B. C. The multipliers are such numbers as: 2, 6, 9, 18, 30, 36, 90, 432, 450, 540, 960, 1080, 2160; see H. V. Hilprecht, "Mathematical, Metrological and Chronological Tablets," Philadelphia, 1906, p. 57–61, 68–69. See also British Museum, "A Guide to the Babylonian and Assyrian Antiquities," third ed., London, 1922, p. 161.

tiplied into such a series of numbers. There are various tablets giving the squares of numbers[7] from 1 to 50, and also the cubes, square roots and cube roots of numbers. But we must be careful not to assume too much from this statement; the tables of square roots and cube roots were really exactly the same as tables of squares and cubes, but differently expressed.[8] In the period we are considering the Egyptian really had nothing to correspond to any of these tables, nor do we know that even the conception of cube root was within his ken. There are also tablets exhibiting the process of division: 60 by its integral factors[9] and 12,960,000 (according to one interpretation) by some of its factors.[10] In connection with one of these latter tablets we find two series of numbers in geometric progression 125, 250 . . . , 16,000; 810, 1620 . . . , 103,680. We shall later refer to another series of numbers in geometric progression 5, 10 . . . , 80 found in an Akkadian tablet of the seventh century B. C. Sumerian multiplication tables would, of course, give numbers in arithmetic progression.

But turning from such tables to arithmetic calling for their application we find that long before coins were in use the custom of paying interest for the loan of produce, or of a certain weight of a precious metal, was common. Sumerian tablets indicate that the rate of interest varied from 20 per cent. to 30 per cent., the higher rate being charged for produce. At a later period the rate was 5½ per cent. to 25 per cent. for metal and 20 per cent. to 33⅓ per cent. for produce.[11] An extraordinary number of tablets show that the Sumerian merchant of 2500 B. C. was familiar with such things as weights and measures, bills, receipts, notes and accounts.

The study of the weights and measures of the Babylonians seems especially important for clearly understanding their mathematics, as Neugebauer, building on work of Thureau-Dangin and others, has recently argued.[12] But time will not permit me to discuss this topic or the large subject of the metrology of the Egyptians.[13]

Let us now consider some characteristics of Egyptian mathematics. We have remarked that tables such as a Babylonian used were not part of an Egyptian's equipment. He did use other tables, however, which dealt with the fundamental feature of his arithmetic, namely, the expression in terms of two or more unit fractions of (*a*) one unit fraction,[14] *e.g.,* $1/7 = 1/14$ $1/21$ $1/42$; (*b*) 2 divided by an odd number,[15] and (*c*) other quotients,[16] as $3 \div 10 = 1/5$ $1/10$; for, with the single exception[17] of 2/3, the Egyptian had no notation for other fractions. After the title page of the Rhind papyrus, more than a quarter of the written part of the papyrus is occupied with a table expressing in this way 2 divided by the various odd numbers 5 to 101 inclusive. For example, $2:7 = 1/4 + 1/28$,

7 For example, in the University of Pennsylvania, dating from about 1300 B. C.; see Hilprecht, "Mathematical, Metrological and Chronological Tablets," Philadelphia, 1906, p. 69, and plate X. See also British Museum, "A Guide to the Babylonian and Assyrian Antiquities," third ed., London, 1922, p. 161.

8 The two tablets with square roots in the University of Pennsylvania date from about 2000 B. C.; see Hilprecht, *ibid*, p. 62–63, and pl. 16. A tablet in the British Museum dating from about 1900 B. C. and containing tables of square roots and cube roots is described in F. Lenormant, "Essai sur un document mathématique chaldéen," Paris, 1868 (for further references in this connection see my Bibliography). For a tablet with squares, cubes, etc., of a given number see British Museum, "A Guide to the Babylonian . . . ," 1922, p. 161. See also M. Cantor, "Babylonische Quadratwurzeln und Kubikwurzeln," *Zeitschrift für Assyriologie,* 21: 110–115, 1908.

To illustrate arithmetical operations of about 2000 B. C. a quotation may be made from a problem on a tablet transcribed by C. Frank, "Strassburger Keilschrifttexte in sumerischer und babylonischer Sprache," *Schriften der Strassburger Wissenschaftlichen Gesellschaft in Heidelberg,* n.s., Berlin, 1928, Heft 9, p. 21: " . . . the square of 13.20 is 2.57.46.40. Add 2.57.46.40 to 50.33.20. It is 53.31.6.40. The square root of 53.31.6.40 is 56.40."

9 A Tello tablet of about 2200 B. C. in Constantinople; no. M. I. O. 7375 in vol. 3 of "Inventaire des tablettes de Tello." (See also L. Delaporte, *Revue d'Assyriologie,* 8: 131–133, 1911.)

10 Four tablets of about 2000 B. C. in the University of Pennsylvania; see Hilprecht, "Mathematical . . . ," p. 61–62, pls. 10, 12, 14, 15. While the tablets exhibit divisions, it is not certain that the dividend is 12,960,000, which Hilprecht called the geometric number of Plato. The literature of this topic is extensive. See, for example, A. G. Laird, "Plato's Geometric Number and the Comment of Proclus," Madison, 1918.

11 M. Jastrow, Jr., "The Civilization of Babylonia and Assyria," Philadelphia, 1915, p. 326, 338; C. H. W. Johns, "Babylonian and Assyrian Laws, Contracts and Letters," New York, 1904, p. 251, 255–256. See also D. E. Smith, "History of Mathematics." vol. 2, 1925, p. 560.

12 O. Neugebauer, "Zur Entstehung des Sexigesimalsystems," Gesellschaft der Wissenschaften zu Göttingen, *Abhandlungen- math.-phys. Klasse,* n.s. vol. 13, no. 1, 1927; Thureau-Dangin, Numération et métrologie sumériennes," *Revue d'Assyriologie,* 18: 123–142, 1921.

13 A source of fundamental importance is F. L. Griffith, "Notes on Egyptian Weights and Measures," Society of Bibl. Archaeology, *Proceedings,* 14: 403–450, 1892; 15: 301–315, 1893.

14 For example in a document of about 1650 B. C., Leather roll, B.M. 10250, *Journal of Egyptian Archaeology,* 13: p. 232–238, 1927, + plates; article by S. R. K. Glanville.

15 As in the Rhind Papyrus of about 1650 B. C.

16 See, for example, the table of the divisions of the numbers 1–9 by 10 in the Rhind papyrus; for a table 2000 years later see H. Thompson, "A Byzantine Table of Fractions," *Ancient Egypt,* 1914, p. 52–54.

17 Compare K. Sethe, "Von Zahlen und Zahlworten bei den alten Ägyptern," (*Schriften der wissenschaftlichen Gesellschaft in Strassburg,* Heft 25), Strasbourg, 1916; it is here made clear (p. 94) that the sign for 2/3 did not, at least in its original form, suggest that 2/3 was thought of as $1/1\frac{1}{2}$.

$2 \div 97 = 1/56 + 1/679 + 1/776$, and both of these results as well as $2/3 = 1/2 + 1/6$ are used in the course of the solution of the thirty-first, of about eighty problems[18] of the papyrus following the table. In these illustrations from the table, 2 divided by an odd number is expressed in terms of the sum of either two or three unit fractions. But four unit fractions are given in eight of the forty-nine divisions. The last of these, $2 \div 101 = 1/101 + 1/202 + 1/303 + 1/606$, is interesting for two reasons: (1) because we first learned of it in 1923 through papyrus fragments discovered in New York City, and (2) because this is the *only* case of this table where a unit fraction has the same denominator as that of the fraction being resolved. All sorts of conjectures have been made, and theories formulated, as to how the author of the papyrus came to select out of the infinite number of possibilities in this table the particular sets given. These discussions include a very elaborate monograph and two doctoral dissertations, one of them appearing less than two months ago at the University of Munich and containing nearly 220 pages.[19]

The form of this table of 2 divided by an odd number is often merely a verification by multiplication and addition of the result stated. For example, in connection with the relation already referred to $2 \div 97 = 1/56 + 1/679 + 1/776$, 97 is multiplied by 1/56 to give 1 1/2 1/8 1/14 1/28; 1/679 of 97 is 1/7; 1/776 of 97 is 1/8. The sum of these products or quotients is 2 as it should be. Now how did the Egyptian arrive at these results? In the case of 1/776 of 97 he would consider what number multiplied by 97 gives 776 and he would multiply 97 successively by 2 thus:

1	97
2	194
4	388
\8	776

from which it follows that the $97 \div 776 = 1/8$. So also for $97 \div 679$; for on adding 4 and 2 and 1 times 97, we find that 7 times 97 gives 679. In the case of $97 \div 56$ the Egyptian would inquire what was the result of taking 1/56 of 97; we may proceed by inquiring what number we would multiply 56 by in order to get 97. The work would be as follows:

\1	56
\1/2	28
1/4	14
\1/8	7
\1/14	4
\1/28	2

from which we see that the multiplier of 56 which gives 97 is 1 1/2 1/8 1/14 1/28. (The check marks which I have used were employed in exactly the same way by the Egyptian.) Let us consider one other example to illustrate further points in multiplication. Suppose the Egyptian were asked to find what parts of 105 loaves will make 82 he might seek multipliers of 105 to give[20] 82. Thus:

1	105
\2/3	70
1/3	35
1/30	3½
\1/15	7
1/10	10½
1/5	21
\1/21	5

Total 2/3 1/15 1/21

To sum up, problems of division are reduced to those of multiplication; multiplication of two numbers is carried through by successive multiplications of one of the numbers by twos or tens or by 2/3, or by divisions of the number by twos or tens. The further extraordinary fact is revealed that in order to get 1/3 of any number, the Egyptian frequently first found the value of 2/3 of the number, and then halved the result. This is illustrated repeatedly in the Rhind papyrus.[21] It would seem as if the Egyptian had specialized in the technique of taking 1/1½ of any number,[22] or used tables for this purpose.[23]

There are only two general rules in the Rhind papyrus; one of these is in problem 61 for finding 2/3 of the reciprocal of an odd number. The rule is as follows: "To get 2/3 of 1/5 take the reciprocals of 2 times 5, and 6 times 5; and in the same way get 2/3 of the reciprocal of any odd number." This rule is

18 For convenience in reference Eisenlohr introduced nos. 1–87 for referring to different parts of the Rhind papyrus, just as Struve uses nos. 1–25 for the Golenishchev papyrus. But numbers 85–87, and some of the others, are not problems.

19 See Lepsius (1865), Loria (1892), Bobynin (1890, 1899), Cantor (1880), Mansion (1888), Hultsch (1895, 1901), Neugebauer (1926), Gillain (1927, 1928), Vogel (1929), and other references in my Bibliography.

20 Compare problems 21 and 22 of the Rhind papyrus.

21 For example, problems 8, 16–20, 25, 29, 32, 38, 42, 43, and 67. See the Chace edition of the Rhind papyrus, vol. I, p. 4, where Gunn's view to the contrary is refuted.

22 That 1/1½ of a number is equivalent to 2/3 of that number is illustrated in problem 33. So also in problem 19 of the Golenishchev papyrus; but possibly this last reference may be quoted in support of Gunn's view that to get 2/3 of a number, 1/3 of that number was first found.

23 It seems probable that such a table was given in a Byzantine table of fractions studied by Thompson, *Ancient Egypt*, 1914; see K. Sethe, "Von Zahlen und Zahlworten bei den alten Ägypten," Strasbourg, 1916, p. 70.

applied several times[24] in the papyrus; the remarkable application in problem 33 gives 2/3 of 1/679 = 1/1358 1/4074.

The second general rule is indicated in connection with problem 66: "If 10 hekat of fat are given out for a year, what is the amount used in a day?" The year is taken as containing 365 days and since there are 320 ro in one hekat, the problem is reduced to finding how many times 365 is contained in 3200 and the answer is found to be 8 2/3 1/10 1/2190 ro. The general rule then follows: "Do the same thing in any example like this."[25]

To illustrate other interesting points let us consider problem 33 of the Rhind papyrus: "A quantity, its 2/3, its 1/2 and its 1/7 added together become 37. What is the quantity?"[26] We would now naturally solve this as a simple equation in algebra $x(1+2/3+1/2+1/7)=37$ and find $x=16\ 2/97$. The Egyptian asked himself what number multiplied into 1 2/3 1/2 1/7 gives 37. By manipulation illustrated above, he soon finds that $16(1+2/3+1/2+1/7)=36\ 2/3\ 1/4\ 1/28$ which is less than the 37 he wants by an amount which he proceeds to determine in using 42 as a comparison number. Taking 2/3 1/4 1/28 of 42, he gets 40 there remains 2; in other words 2/3 1/4 1/28 differs from 1, which was necessary to add to 36 in order to give 37, by 1/21. The author then shows that 1 2/3 1/2 1/7 applied to 42 gives 97 and hence knows that 1/97 of 1 2/3 1/2 1/7 is 1/42; whence, multiplying this result by 2, he gets $2 \div 97$ or 1/56 1/679 1/776 of 1 2/3 1/2 1/7 is equal to 1/21. Hence he has as answer 16 1/56 1/679 1/776.

By my method of presentation I have tried to indicate that in use of a comparison number 42 the Egyptian did not necessarily think of fractions other than unit fractions, although we might say that his operation was *equivalent to* bringing to a common denominator. The question comes up again in vastly more complicated form when he proves that the result he has got is correct. After writing down the products

1	16 1/56 1/679 1/776
2/3	10 2/3 1/84 1/1358 1/4074 1/1164
1/2	8 1/112 1/1358 1/1552
1/7	2 1/4 1/28 1/392 1/4753 1/5432

he proceeds to prove that the sum of his products is 37 and makes use of the comparison number 5432.

The question as to whether the Egyptian had a *conception* of the general fraction has been the topic of much discussion. Sethe believed that he did not;[27] but Peet[28] and Wieleitner[29] hold to a contrary view. By excessively brief suggestion I have attempted to present the somewhat original view set forth in the Chace edition of the Rhind papyrus.

One is tempted to dwell on some other Egyptian problems which we might solve by simultaneous equations. Turning to the Rhind papyrus, we may consider problem 40, which we would naturally solve with two linear simultaneous equations. The problem is as follows: "Divide 100 loaves among 5 men in such a way that the shares received shall be in arithmetical progression and that 1/7 of the sum of the largest three shares shall be equal to the sum of the smallest two. What are the shares?" The solution starts in at once: "Do it thus: Make the difference of the shares 5 1/2. Then the amounts the five men receive will be 23 17½ 12 6½ 1, total 60. As many times as it is necessary to multiply 60 to make 100, so many times must these terms be multiplied to make the true series, and it is found that 1 2/3 times 60 make 100 and hence the shares 38 1/3, 29 1/6, 20, 10 2/3, 1 2/3. The use of proportion here is especially interesting.[30] For the very attractive method by which the Egyptian probably arrived at the difference 5 1/2 on assuming that one man's share was 1, see Chace's edition of the papyrus. The problem is notable also as illustrating that while the Egyptian did not consider such things as simultaneous equations, he was nevertheless able in certain cases to obtain the result by methods at his command.

Problem 64 is another one involving an arithmetic progression[31]: "Distribute 10 hekat of barley among 10 men in such a way that the shares shall be in arithmetical progression with a common difference of 1/8 hekat. What is the share of each?"[32]

Problem 79 seems to be as follows: "In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, each ear of spelt will produce 7 hekat of grain; how much grain is thereby saved?" But the author gives only the sum of the five terms of the geometric progression of which the first term is 7 and the multiplier 7. The

24 Rhind Papyrus, problems 17, 30, 33, 61.

25 Compare the Latin, "*Fac ita,*" and the early German, "*Thu ihm also.*"

26 This problem is exactly the same as problem 31, if 33 is substituted for 37.

27 K. Sethe, (*a*) "Von Zahlen und Zahlworten bei den alten Ägyptern," Strasbourg, 1916, p. 62; (*b*) "Deutsche Mathematiker-Vereinigung," *Jahresbericht,* 33: 141, 1925.

28 T. E. Peet, "Rhind Papyrus," p. 16–17.

29 H. Wieleitner, "Kannten die Ägypter den Begriff eines allgemeien Bruches?" *Mitteilungen zur Geschichte der Medizin und der Naturwissenschaften,* 25: 1–4, 1926.

30 Problem no. 7 of the Golenishchev papyrus contains a special word for ratio; compare Peet, p. 60.

31 A third Egyptian problem involving an arithmetic progression is in the Kahun Papyri of about 1850 B. C.; see F. L. Griffith, "The Petrie Papyri. Hieratic Papyri from Kahun," 2 vols., London, 1897–98 (see Bibliography).

32 I have given a new solution of this problem in *Isis,* December, 1928, 11: p. 397 (in line 12 of this page *for* "shares 3" *read* "share,").

proof which he gives has been the subject of speculation as to whether the Egyptian knew of a certain general relation in connection with geometric series; but it has recently become clear[33] that nothing but multiplication is here involved. The problem is of interest from another point of view. In a thirteenth century work of Leonardo of Pisa the following problem occurs[34]: "7 old women invaded Rome; each woman had 7 donkeys; each donkey carried 7 sacks; each sack contained 7 loaves of bread; with each loaf were 7 knives; each knife was in 7 sheaths? What is the total?" The persistence of this type of problem after twenty-eight centuries is notable. Even in our own day we have: "As I was going to St. Ives I met a man with 7 wives; each wife had 7 sacks; each sack had 7 cats; each cat had 7 kits; kits, cats, sacks and wives, how many were there going to St. Ives?"

No sketch of Egyptian mathematics should fail to refer to the process of false position which was so freely used[35] and which has been employed through the centuries even down to our arithmetics of a generation ago. The method consists in assuming a certain numerical answer and then, by performing the operations of the problem, arriving at a number, which can be compared with a given number, the true answer having the same relation to the assumed answer that the given number has to the number thus obtained. Consider an example. In problem 27 it is required to find a quantity such that it and its 1/5 added together become 21. In the solution 5 is assumed for the quantity then the quantity and its 1/5 make 6. As many times as 6 must be multiplied to give 21, so many times must 5 be multiplied to give the required number. It is found that 6 had to be multiplied by 3½; hence the required number is 5 multiplied by 3½ or 17½.

In concluding these references to various arithmetic operations and processes it should be noted that the idea of squaring a quantity was not unknown to the Egyptian. This is indicated by its use in problems 11 and 14 of the Golenishchev papyrus. So also for the square root of a number, for which a special sign is used in problems 6, 7 and 17 of the Golenishchev papyrus, in one of the Kahun papyri,[36] and in Berlin papyrus 6619,[37] all three dating from about 1850 B. C. The square roots of integers are found in all but the two examples of the Berlin papyrus, where the square roots of 6 1/4 and of 1 1/2 1/16 are given.

But there are other problems not so interesting for their mathematics as for other things. For example, problem 67 of the Rhind papyrus starts out in this way: "The herdsman came to the stock-taking with 70 cattle. The accountant said to the herdsman, Very few tribute-cattle art thou bringing; pray where are all thy tribute cattle? The herdsman replied to him, What I have brought is 2/3 of 1/3 of the cattle that thou hast committed to my care. Count and thou wilt find the full number." In this way it appears that 70 out of 315 cattle were levied by the owner as tribute.

Problem 62 is as follows: "Example of reckoning the contents of a bag of various precious metals. Suppose it is said to thee, A bag containing equal weights of gold, silver and lead has been bought for 84 rings. What is the amount in it of each precious metal, that which is given for a deben of gold being 12 rings, for a deben of silver 6 rings and for a deben of lead 3 rings?"

The feed for geese, cranes, ducks, quails, doves, also for bulls and common cattle, is discussed in problems 82–84. In the Golenishchev papyrus is the following rule-of-three: Problem 23: "A sandal maker works for 15 days receiving wages every five days. If he does the work in 10 days after what periods should he be paid?"

Then, too, 10 of the 25 problems in the Golenishchev[38] and 10 of about 80 in the Rhind [39] papyrus are so-called pefsu, or cooking ratio, problems. Pefsu is the number of units of food or drink that could be made from a unit of material in the process of cooking, and it determined the relative value of any food or drink; the lower the pefsu, the more valuable the unit of food. A couple of enunciations only may be given in illustration: (*a*) Rhind, no. 69, "3½ hekat of meal are made into 80 loaves of bread. Let me know the amount of meal in each loaf and what is the pefsu." (*b*) Rhind, 78, "Suppose it is said to thee, 100 loaves of pefsu 10 are to be exchanged for a quantity of beer of pefsu 2. How many des of beer will there be?"

Any survey of the mathematics of the Egyptians ought to include at least a brief reference to one of their extraordinary monuments, the great pyramid at Gizeh, erected during the reign of Cheops about 2900 B. C. The mere fact that the erection of such a structure was possible indicates a very remarkable governmental and social organization. It is said that 100,000 workmen were kept constantly employed on this structure for fifty years, ten years of this period being used in constructing a road to the limestone quarry some miles distant. It has been estimated

[33] O. Neugebauer, "Die Grundlagen der ägyptischen Bruchrechnung," Berlin, 1926, p. 14–15.

[34] Leonardo of Pisa, "Scritti," vol. 1, Rome, 1857, p. 311–312.

[35] For example in problems 24–27, 35, 37, 38 of the Rhind papyrus; also in 40 which we have already discussed.

[36] F. L. Griffith, "The Petrie Papyri. Hieratic Papyri from Kahun and Gurob," vol. 2, plate VIII.

[37] H. Schack-Schackenburg, *Zeitschrift für ägyptische Sprache,* 38: 138, 1900; 39: 65, 1901.

[38] Nos. 5, 8, 9, 12, 13, 15, 16, 20, 22, and 24.

[39] Nos. 69–78.

that over this road were brought 2,300,000 blocks of stone averaging two and one half tons in weight. These blocks were fitted together very perfectly. For the roofs of chambers granite blocks twenty-seven feet long, six feet high and four feet thick, and weighing some fifty-four tons each, were brought from the quarry over 600 miles away and conveyed by a brick ramp to their position over 200 feet above the ground level. The pyramid covered about thirteen acres and recent surveys show[40] that its base was almost a perfect square, no side differing from the mean length of 755.78 feet by more than 4½ inches, while two of its angles differ from 90° by less than 34 seconds, and the other two by less than 3½ minutes.[41] The orientation of a side of the square being almost exactly north and south was an indication not only of careful observations of the stars by the Egyptians but also of astrological considerations, so prevalent in those days. The height of the pyramid was about 481 feet, and there were various passage ways to the chambers within the structure. Especially since the publications of John Taylor in 1859 and 1864, and of C. Piazzi Smyth, astronomer royal for Scotland, in 1864 and 1867, even down to 1929, all sorts of curious pyramid mysticism has been the subject of discussion in scores of chapters, articles, pamphlets and books.[42] In connection with the pyramid of Gizeh such mysticism has led to conclusions like the following being regarded as gospel: (1) it was built in the present proportions so that the perimeter of its base should be exactly equal to the perimeter of a circle of which the pyramid's height was radius; (2) the perimeter of the base was so chosen that it might contain one hundred times as many pyramid inches as there are days in the year; (3) the pyramid inch is equal to a five hundred millionth part of the earth's polar diameter; (4) the total area is divided in golden section, that is, the area of the base is to the sum of the areas of the faces as this sum is to the sum of the areas of the faces and base; (5) it was intentionally placed exactly in latitude 30°.

Had De Morgan lived a few years longer his recreation of crushing circle-squarers might well have been set aside for plaguing pyramiders.

But the problems of mechanics and engineering involved in handling even the larger stone blocks of the pyramids were slight as compared with those dealt with by Egyptians in quarrying and setting up some of their huge obelisks[43] of pink granite. The largest existing obelisk, quarried about 1500 B. C., was no less than 105 feet long, nearly ten feet square at the larger end and about 430 tons in weight. It was set up in front of the Temple of the Sun at Thebes, but moved, about 1,800 years later, to the piazza of St. John Lateran at Rome, where it may be seen to-day.

But another extraordinary fact to be noted, concerning the engineers or surveyors contemporary with the pyramid and obelisk builders, is that they were already in possession of methods for laying out nilometers around innumerable bends in the river Nile for a distance of seven hundred miles such that the zero points of the nilometers, always below lowest water, are all in one plane.[44]

Let us now turn from applied to results which we usually associate with pure mathematics and consider the geometry of the Babylonians and Egyptians.

BABYLONIAN GEOMETRY

One of the most interesting of Babylonian tablets was found at Tello, Arabia, and is now in the Ottoman Museum at Constantinople. It dates back to about 2200 B. C. and was first described in 1896 by Eisenlohr[45] and Oppert,[46] and later by Thureau-Dangin.[47] It is the plan of a great field divided into 15 parts: 7 right triangles, 4 rectangles (nearly) and 4 trapezia[48] (one side always perpendicular to the parallel sides). The lengths of the lines and the areas of the parts which are indicated show that the following geometrical results, probably derived wholly empirically, were known at that time:

1. The area of a rectangle is the product of the lengths of two adjacent sides.

2. The area of a right triangle is equal to one half the product of the lengths of the sides about the right angle.

3. The area of a trapezium with one side perpendicular to the parallel sides is one half the product of the

40 I. H. Cole, "Determination of the Exact Size and Orientation of the Great Pyramid of Giza," Survey of Egypt, Paper no. 39, Cairo, 1925; L. Borchardt, "Längen und Richtungen der vier Grundkanten der grossen Pyramide bei Gise," Berlin, 1926.

41 The results quoted by J. H. Breasted in this connection (*Scientific Monthly*, 10: 92, 1920) have since been found to be incorrect.

42 Particularly under 1854, 1910, and 1928 (Suppl) in my Bibliography are some sample titles in this field. See also Borchardt's pamphlet listed under 1922.

43 A recent valuable work devoting considerable space to the mechanics of setting up obelisks is "The Problem of the Obelisks from a Study of the Unfinished Obelisk at Aswan," London, 1923, by R. Engelbach, an Englishman who is director of the Egyptian Museum at Cairo.

44 J. H. Breasted, *Scientific Monthly*, 10: 91, 1920. See also L. Borchardt, "Nilmesser und Nilstandsmarken," Berlin, 1906 (in Preuss. Akad. d. Wissenschaften, *Abhandlungen*).

45 A. Eisenlohr, *Ein altbabylonischer Feldplan*, Leipzig, 1896.

46 J. Oppert, Académie d. Inscriptions et Belles-Lettres, *Comptes Rendus*, s. 4, v. 24, 1896, p. 331–348; also in *Revue d'Assyriologie et d'Archéologie Orientale*, v. 4, 1897, p. 28–33.

47 F. Thureau-Dangin, *Revue d'Assyriologie*, v. 4, 1897, p. 13–27.

48 The meaning of the terms "trapezium" and "trapezia" of this paper is that used in every country of the world except the United States.

length of this perpendicular and the sum of the lengths of the parallel sides.

For the history of the Pythagorean theorem a portion of an Akkadean tablet in the Prussian State Museum, dating back to about 2000 B. C., is of special interest. It was published by Weidner[49] in 1916. The figure of a rectangle with one diagonal is drawn and the dimensions of the rectangle, 10, "breadth," and 40, "height," are given. Two methods are used to calculate the length c of the diagonal. The first method leads in numbers to the approximation $c = a + 2ab^2.60^{-2}$, where a is the length of the greater side and b of the lesser. The text of the solution here translated freely is as follows:

Square the side of length 10 then thou will get $1 \cdot 40$ $[60 + 40 = 100]$. The square area 1.40 [100] multiplied by the length 40 of the other side gives thee 1.6.40 $[1 \cdot 60^2 + 6 \cdot 60 + 40 = 4000]$. In doubling thou wilt get 2.13.20 $[2 \times 62^2 + 13 \cdot 20 = 8000]$ adding that to 40 thou wilt get 42.13.20 $[42 + 13 \cdot 60^{-1} + 20 \cdot 60^{-2}]$ as diagonal. Such is the calculation.

The second method seems to lead in numbers to the relation $c = a + b^2/2a$ which is what one arrives at in the calculation of $(a^2 + b^2)^{1/2}$ if terms after the second are neglected. The result found here is $41 + 15.60^{-1}$. How were the Akkadians led to such formulas? It is not possible to believe that in the case of the second approximation they thought of anything like a binomial expansion which we find natural, even though Hilprecht be correct in thinking that they were familiar with the expansion[50] $(a+b)^2 = a^2 + 2ab + b^2$. Rather are such approximations remarkable developments to meet needs, as we observe in other directions, in connection with these extraordinary people.

This tablet suggests that the Babylonians may have known the Pythagorean theorem for a right triangle. This appears to be a certainty when we consider two among forty-seven mathematical problems in *CT*.IX. It was less than nine months ago, in a paper by Struve and Neugebauer,[51] that the meaning of these two problems became clear. In the first of these problems we are given the length 60 of the circumference of a circle and the length 2 of a perpendicular from the center of a chord of the circle to the circumference; it is required to find the length of the chord. If d denotes the length of the diameter of a circle, s the length of a chord and a the height of the arc corresponding to this chord then

$$s = [d^2 - (d - 2a)^2]^{1/2}.$$

The various operations in the solution of the problem seem definitely to prove that the equivalent of this relation was familiar to the Babylonians of 2000 B. C. The text is as follows:

1 [=60] the circumference, 2 the perpendicular, to find the chord. Do as follows, square 2, thou seest it is 4. Subtract 4 from the diameter 20 thou seest it is 16. Square the diameter 20 thou seest it is 6.40 [=400]. Square 16 thou seest it is 4.16 [=256]. Subtract 4.16 from 6.40 giving 2.24 [=144] of which calculate the square root. The square root is 12. Such is the procedure.

This solution brings out, what may be checked in other parts of *CT*.IX, that 3 was the value of π used by the Babylonians. It will be recalled that this was the value we find in the Bible, for example, in I Kings 7: 23, which was written about 560 B. C. and may have been taken from temple records dating back to 900 B. C. This verse is as follows: "And he made a molten sea, 10 cubits from one brim to the other: it was round all about, and his height was 5 cubits and a line of 30 cubits did compass it round about." We note also that the proposition of Thales of Miletus that the angle in a semi-circle is a right angle was visualized by the Babylonians 1,400 years earlier, and that a study was made of chords of a circle long before the days of Hipparchus.

The second problem in *CT*.IX to which we have referred is to find a given $d = 20$ and $s = 12$; each step is equivalent to substitution in the formula

$$a = 1/2[d - (d^2 - s^2)^{1/2}].$$

Other results in *CT*.IX are: (a) The area of a circle is one twelfth the square of the length of its circumference; (b) the volume of the frustum of a cone is one half the sum of the areas of the circular bases times the distance between them.

Another series of problems of great mathematical interest is found on tablets of about 2000 B.C. in the library of the University of Strasbourg. These were published with a translation in 1928 by Carl Frank[52] and their meaning was ingeniously elaborated this year by Neugebauer.[53] The problems treat of various

[49] E. F. Weidner, "Die Berechnung rechtwinkliger Dreiecke bei den Akkadern um 2000 v. Chr.," *Orientalestische Literaturzeitung*, v. 19, 1916, cols. 257–263; see also commentary by A. Ungnad, cols. 363–368, and by H. Zimmern, cols. 321–325. A reference may be given also to O. Neugebauer, "Zur Geschichte des Pythagoräischen Lehrsatzes," Gesellschaft der Wiss. zu Göttingen, *Math.-phys. Kl., Nachrichten*, 1928, p. 45–48.

[50] H. V. Hilprecht, "Mathematical, Metrological and Chronological Tablets from the Temple Library of Nippur," Philadelphia, 1906, p. 24.

[51] O. Neugebauer and W. Struve, "Über die Geometrie des Kreises in Babylonien," *Quellen und Studien zur Geschichte der Mathematik, Abteilung B: Studien*, Berlin, 1929, p. 89–92.

[52] C. Frank, "Strassburger Keilschrift Texte in sumerischer und babylonischer Sprache," Strassburger Wissenschaftlichen Gesellschaft in Heidelberg, *Schriften*, n.s., part 9, Berlin and Leipzig, 1928.

[53] O. Neugebauer, "Zur Geschichte der babylonischen Mathematik," *Quellen und Studien zur Geschichte der Mathematik, Abt. B, Studien*, v. 1, 1929, p. 67–80.

sections of a right triangle by lines drawn parallel to the base, which is regarded as horizontal, the vertex being below the base. In problem 10, two lines are drawn parallel to the base and the figure consisting of the two adjacent trapezia is considered. The area of the upper trapezium is given as 783, of the lower as 1,377. The difference of the base (b) and the lower dividing line (d_2) is given as 36. It is also given that the difference between the base and the upper dividing line (d_1) is one third of the difference between the upper and lower dividing line. It is required to find five unknown quantities: the lengths of the base and of the two dividing lines, and the altitudes of the trapezia (h_1 and h_2). The various statements above lead us to the following relations: $(1/2)h_1(d_1+b)=783$; $(1/2)h_2(d_2+d_1)=1377$, $b-d_2=36$, $b-d_1=(1/3)(d_1+d_2)$. It follows from this last equation and from similar triangles that $h_2=3h_1$, and from these five relations between five unknown quantities it is found that $b=48$, $d_1=39$, $d_2=12$, $h_1=18$, $h_2=54$. The mere formulation of such a problem gives a striking impression of the mathematical capabilities of the Babylonians. Neugebauer seems to have thought that as a result of this problem we might say that they knew how to solve five equations in five unknown quantities. To me such a suggestion is highly misleading; the much more natural deduction is that the Babylonians had developed considerable insight into geometrical relations (further examples of this might be shown[54] if time permitted), and solved the problem entirely from geometrical considerations. This would imply, in particular, that they were familiar with the result that similar right triangles have the sides about the right angles proportional, a theorem which our mathematical histories connect with Thales.

In another problem a right triangle is divided into five parts by four lines drawn parallel to the base, and from certain data other parts are to be found. In yet another, a triangle is divided into two parts (a trapezium and a triangle) by a single dividing line (of length d) parallel to the base, which is given as 30 units in length. The altitude of the triangle is given as 10 units greater than h_1, the altitude of the trapezium, and the area of the triangle is given as 270. Whence $270 = (1/2)d(h+10)$; making use of the result regarding similar right triangles referred to above, we would have also $d:(h+10)=30:(2h+10)$ from which, according to methods of to-day, we would be naturally led to a quadratic equation for determining h. Neugebauer notes two other similar problems the discussion of which leads to quadratic equations whose solution is called for by the problem. It is exactly in connection with these questions, however, that details as to the solutions of the questions are lacking. Were it not for a note at the end of the article we might incline to regard the introduction of quadratic equations as out of keeping with the mathematics discussed. In this note, however, Neugebauer states that in another problem of *CT*.IX the discussion leads to the quadratic equation

$$x^2-\frac{2F}{d}x+\frac{2Fh}{d}=0$$

and that the solution follows step for step by substitution in the formula:

$$x=\frac{F}{d}-\left[\left(\frac{F}{d}\right)-\frac{2Fh}{d}\right]^{1/2}.$$

In an article published last August Wieleitner reported[55] that he had inspected an unpublished manuscript of Neugebauer and was thoroughly convinced by the evidence adduced that the Sumerians of 2000 B. C. understood how to solve problems which led to a general quadratic equation—a new fact of very extraordinary interest. It was previously supposed that Heron of Alexandria who flourished about 250 A. D. was the first to use our modern method of solving a quadratic equation.[56]

Although more a matter connected with astronomy, we may possibly under the head of geometry consider the Babylonian divisions of a circle. Bosanquet and Sayce stated[57] with great definiteness that the divisions which are found are those into 8, 12, 120, 240 and 480 parts; that we do *not* find the division into 360 parts as commonly supposed; and that while the division of the circle as practiced by Ptolemy and in modern times is an outgrowth of the sexagesimal method of the inscriptions, the latter does not contain the former. In support of his argument the authors quote an Akkadian tablet[58] in the British

[54] For example, see C. J. Gadd, "Forms and Colours," *Revue d'Assyriologie et d'Archéologie Orientale*, 19: 149–158, 1922. The article is mainly descriptive of a British Museum tablet of about 2000 B. C. on which are drawn about a dozen figures involving squares and arcs of circles. At least two of the figures suggest either the study of various geometrical forms for their own sake or the application of such designs in artistic ornamentation.

[55] *Archiv für Geschichte der Mathematik, der Naturwissenschaften, und der Technik*, 12: 107.

[56] T. L. Heath, "History of Greek Mathematics," Oxford, 1921, vol. 2, p. 306, 344.

[57] R. H. M. Bosanquet and A. H. Sayce, "The Babylonian Astronomy," Royal Astronomical Society, *Monthly Notices*, 40: 108–109, 1880.

[58] First published, in part, by E. Hincks, in *Literary Gazette*, 38: 707, 1854, who surmised that the numbers referred to the portions of the moon visible on each of the first fifteen days of the month. This view was repeated by Hincks in Royal Irish Academy, *Polite Literature, Transactions*, 22: 407, 1855, and repeated by Cantor, Tropfke and Karpinski (see my Bibliography under Hincks, 1854 suppl.). They seem to have overlooked the correct interpretation.

Museum containing a record of the moon's longitude during thirty days; these numbers during the first fifteen days of its advance are: 5, 10, 20, 40, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240; then follow retrograde numbers 224, 208, 192, etc., the moon's daily motion being, for the most part, 16 out of 480 divisions of a circle, as it should be roughly. In a tablet from Sennacherib's palace (about 700 B. C.), now in the British Museum, a circle is divided into 480 equal parts.[59]

On the other hand, in discussing geometry of the Babylonians, Cantor states:[60] "for a certainty we have the division of a circle into 6 parts, then into 360 degrees." Heath's discussion of the question[61] involves no such positive statement, but he does note that it was Hipparchus, about 50 B. C., who first divided the circle in general into 360 parts or degrees, and that the introduction of this division coincides with his invention of trigonometry. In an article published by Thureau-Dangin last year[62] it was argued that the division of the circle into 360 parts by the Babylonians was "natural," but that further sexagesimal division was unnatural; also that the Babylonians made use of the division of the circle as learned from the Sumerians. It has appeared to me that the evidence favors the view of Bosanquet and Sayce that the division of the circle into 360 parts did *not* originate with the Babylonians.[63]

And finally, in connection with Babylonian geometry, there does not seem to be any problem where an expression for volume is clearly stated. It is true that Hilprecht discussed such an example but he was in doubt as to whether the object was a rectangular parallelopiped or not.

EGYPTIAN GEOMETRY

Let us first consider the problems of a geometrical nature, found chiefly in the Rhind and Golenishchev papyri. In the Rhind papyrus there are nineteen problems of this kind, namely: nos. 41–46, 48–60. The first six problems deal with volumes and the next eight with areas. In the first three are found volumes of circular cylindrical granaries of different diameters and heights. In each the process consists in multiplying the height by the area of the base which is found by the uniform rule of squaring eight ninths of the diameter of the base. This leads us to the value $256/81 = 3.1605 \ldots$ for π, a very remarkable approximation, used also in the Golenishchev papyrus and therefore practically contemporary with the Babylonian use of $\pi = 3$, about 2000 B. C. This same method of discussing the area of a circle is found in problem no. 48, where the areas of a circle and its circumscribed square are compared, and in no. 50, where the area of a round field of given diameter is asked for.

Among various writers who have conjectured how the Egyptian arrived at such a rule for the area of a circle are Demme, Simon and Vacca. In essence Simon's conjecture, which seems the most plausible, amounts to the following:[64] Suppose that two vessels of the same height, one with a circular base and the other with a square base equal to the circumscribed square of the circle, are filled with water. Then by weighing the water in each container it would be found that the quantities are as 64 to 81. Hence the result of problem no. 48 with reference to the relative areas of a circle and its circumscribed square would be found, and the rule for finding the area of a circle derived.

In problems 44 to 46 are discussed volumes and dimensions of parallelopipedal granaries. In other problems the area of a rectangular field of given dimensions is found, sections of a triangle similar to those considered by the Babylonians are discussed and two problems of the following type are solved: What equal areas should be taken from 10 fields if the sum of these areas is to be 7 setat?

Problem 51 is to find the area of a triangle given its base and altitude and 52 to find the area of a trapezium with apparently equal slanting sides, the lengths of the parallel sides and the distance between them being given numbers. These two problems have been the topic of many pages of discussion during the past fifty-three years. Some of the questions involved may be briefly set forth. In no. 51 the figure is a fairly good approximation to an isosceles triangle; beside the base and one of the sides are written certain numbers, half the product of which is given as the area of a triangle. Did the author of the papyrus think that the area of an isosceles triangle was one half the length of the base times the length of a side? Or was this triangle intended to be regarded as right angled and not isosceles? And, similarly in problem 52, did the author think that the area of the trapezium is one half the sum of the lengths of the parallel sides times the length of a

[59] D. E. Smith, "History of Mathematics," vol. 2, Boston, 1925, p. 230.

[60] M. Cantor, "Vorlesungen über Geschichte der Mathematik" vol. 1, 3rd ed., Leipzig, 1907, p. 50.

[61] T. L. Heath, "History of Greek Mathematics," vol. 2, 1921, p. 214–216.

[62] *Revue d'Assyriologie et d'Archéologie Orientale*, 25: 187–88, 1928.

[63] The weighty authority of B. Meissner ("Babylonien und Assyrien," vol. 2, Heidelberg, 1925, p. 385) favors the 360 division of the circle by the Babylonians; but the statement is made in such a way as to suggest that Meissner did not consider the question as carefully as he might have done.

[64] M. Simon, *Archiv d. Mathematik und Physik*, s.3, 9: 102, 1905.

slant height? Or was one side of the trapezium intended to be perpendicular to the parallel sides as in the case of those studied by the Babylonians?

In the great dedicatory inscription, of about 100 B. C., in the Temple at Edfu, as described by Lepsius, there are references to a large number of four-sided fields. For each of these the lengths of the sides (which we may call, in order as we go around, a, b, c, d) and their areas are given; these areas may be determined by the formula $(1/2)(a+c) \times (1/2)(b+d)$. If $a=c$, and b and d are parallel, we see that the Egyptians of 100 B. C. *did* use the formula that the area of a trapezium is equal to one half the sum of the parallel sides times the slant height, when these are equal. Moreover, if $a=c$, $d=0$, we get $\frac{1}{2}ab$ as the area of an isosceles triangle which is illustrated on the Edfu inscription by two examples $a=c=17$, $b=5$, area 85/8; $a=c=3$, $b=2$, area 3. These rules for calculating the areas of four-sided and three-sided fields are precisely those used by Egyptian natives to-day.[65]

It is, therefore, all the more interesting that in the period which we are considering the correct formulas were used for the areas of a triangle and a trapezium. The final proof of this, even when the figures are not isosceles, is given in papers published by Gunn and Peet in 1926 and last month.[66] The proof is based upon the analysis of the meaning of certain Egyptian words and of problem 4 of the Golenishchev, which is identical with no. 51 of the Rhind papyrus.

Five of the six remaining geometrical problems in the Rhind papyrus deal with the relation of the lengths of two sides of a right triangle which corresponds to the cotangent of the angle which a face of a regular pyramid makes with its base. This is called the *seked* of the pyramid. The seked 18/25 in problem 56 has been associated with the slope of the lower half of the southern stone pyramid of Dahshur, and the seked 3/4 in problems 57–59 with the slope of the second pyramid of Gizeh. The absurdities of such associations, even as made by Heath, I have recently shown elsewhere.[67]

Of six geometrical problems in the Golenishchev papyrus we have already noted the identity of no. 4 with a problem of the Rhind papyrus. In no. 6 we are given a rectangular enclosure of 12 units area and the ratio of the sides as 1: 3/4 [1/2 1/4]; the lengths of the sides are found. The problem seems to be practically identical with one in the contemporary Kahun papyri in London. The solution of this problem may be regarded as equivalent to solving the simultaneous equation $xy=12$, $x:y=3/4$ [$-$1/2 1/4]. For the steps of the solutions in both papyri are as follows:

$$1: 3/4 = 1\ 1/3;\ 1\ 1/3 \cdot 12 = 16;\ (16)^{1/2} = 4\ [=x];\ 4 \cdot 3/4 = 3\ [=y].$$

Two other geometrical problems in Berlin papyri fragments of about 1850 B. C. may be said to lead to simultaneous quadratic equations which are solved by the method of false position. The first of these is as follows:

> Distribute 100 square ells between two squares whose sides are in the ratio 1 to 3/4; the corresponding equations[68] are $x^2+y^2=100$, $x:y=1:3/4$. The solution is as follows: Try $x=1$, $y=3/4$, then $x^2+y^2=25/16$ [$=1$ 1/2 1/16]. But $(25/16)^{1/2}=5/4$ [$=1$ 1/4] and $(100)^{1/2}=10$. $10:5/4=8$, whence[69] $x=1\cdot 8=8$, $y=6$.

But while there is equivalence between the solutions of these ancient Egyptian problems and the solution of our algebraic simultaneous equations, so far as we know the Egyptian had none of our modern algebraic conceptions in this connection; he used methods which are wholly arithmetic, the idea of ratio and proportion being an important element. Cantor[70] and Tropfke[71] seem to be somewhat misleading in this regard.

The correct interpretation for no. 7 of the Golenishchev papyrus was given only last month by Gunn and Peet. It is: "A triangle of given area is such that its altitude is $2\frac{1}{2}$ times its base; find both." The authors make clear that no. 17 is a similar problem for a scalene triangle.

But problems 14 and 10 of the Golenishchev papyrus contain the most extraordinary results in ancient geometry. In the first of these the conclusion is unavoidable that the Egyptian of about 1850 B. C. was

65 B. Gunn, *Journal of Egyptian Archaeology*, 12: 133, 1926.

66 B. Gunn, *Journal of Egyptian Archaeology*, 12: 132–133, April, 1926. B. Gunn and T. E. Peet, *Journal of Egyptian Archaeology*, 15: 173–176, November, 1929.

67 In my Bibliography.

68 For the second problem, the equations are: $x^2+y^2=400$ and $x:y=2:1\frac{1}{2}$.

69 We may note the relation in this problem of $1^2+(3/4)^2=(5/4)^2$, which has $3^2+4^2=5^2$ as a basis. Contrary to what is popularly believed, there is nothing to prove that the Egyptian knew the theorem regarding the squares on the sides of right triangles even in special cases.

70 M. Cantor, "Vorlesungen über Geschichte der Mathematik," vol. 1, third ed., 1907, p. 95–96. It is true that Cantor treats these problems under the head of geometry and after his discussion states (p. 96): "Wir möchten bitten diese ganze Untersuchung, welche ihrem algebraischen Inhalte nach schon in das vorige Kapitel [arithmetic and algebra] gehören könnte, nicht als hier an unrichtiger Platze stehend bemängeln zu wollen. Sind doch die behandelten quadratischen Gleichungen aus geometrischen Aufgaben entsprungen."

71 Tropfke, "Geschichte der Elementar-Mathematik," second ed., vol. 3, 1922, p. 56–57; these pages are in Tropfke's history of equations of the second degree.

familiar with our formula for the volume of a frustum of a square pyramid $V=\frac{h}{3}(a_1^2+a_1a_2+a_2^2)$, where a_1 and a_2 are the sides of the square bases of the frustum, and h its height. Previous to 1917 it was not known that any particular case of the familiar general formula for the volume of the frustum of a pyramid given by Leonardo of Pisa in 1220 had been previously used. To learn then[72] that such a formula was known 3,000 years earlier was sufficient almost to revolutionize our ideas concerning the capability of the Egyptian as a geometer. Of course Heron of Alexandria, Brahmagupta and Mahāvīrācārya gave other forms for the volume of the frustum of a square pyramid. The exact text in this connection may be given:[73]

Example of calculating a truncated pyramid. If you are told: a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top:

You are to square this 4; result 16. You are to double 4; result 8. You are to square this 2; result 4. You are to add the 16 and the 8 and the 4; result 28. You are to take 1/3 of 6; result 2. You are to take 28 twice; result 56. See, it is of 56.

You will find (it) right.

In a joint article, published last month by Gunn, Peet and Engelbach, detailed consideration was given to the question of how the Egyptian arrived at such a formula,[74] but to me their conclusions are not very convincing, even though we assume the great stage of their mathematical development which led them to the result of problem 10 which seems to be even more remarkable than the one just discussed.

In this problem taking the value of π used in the Rhind papyrus, the area of a hemisphere is found correctly, a result till recently supposed unknown before the time of Archimedes 1,600 years later. If d be the diameter of a hemisphere, each step of the work is equivalent to substitution in the formula:

$$S=[(2d-2d/9)-(1/9)\ (2d-2d/9)]\cdot d$$

which may be simplified to give $S=(1/2)(256/81)d^2$.

In concluding the survey of Egyptian geometry, reference may be made to a theory of F. G. Röber[75] who, from the study of ancient temple architecture and of the Temple of Edfu in particular, seemed to have divined that the layout was intimately connected with the construction of a regular heptagon. Over the ruler and compasses construction for this heptagon indicated by Egyptian architecture Sir William Rowan Hamilton grew highly enthusiastic; he showed[76] that the cosine of the angle subtended by a side of the heptagon thus constructed agreed to within two units in the seventh place of decimals with the value for the cosine of the seventh part of four right angles.

CONCLUDING COMMENT

Time will not allow me further to elaborate my topic. We have obtained some glimpses of the extraordinary mathematical achievements of the Egyptians during the golden period of their activity between 1600 and 1900 B. C. So far as we know at present practically no advance was made by them at a later date. We have met with some results, and many others could be adduced, which seem conclusively to show that the Egyptian studied mathematics for its own sake, and not alone for its practical applications. Breasted was led to make a similar deduction in connection with medical science after studying the remarkable Edwin Smith medical papyrus, soon to be published, which is also a document of the golden period of mathematical activity. What does the future have in store for the extension of our knowledge of Egyptian mathematics? The Golenishchev papyrus is to be published completely in the near future, but there are no other known unpublished Egyptian mathematical documents of this ancient period.[77]

We have seen that even our present knowledge of Babylonian mathematics indicates notable achievements, and suggests that they too must have studied mathematics for its own sake. But since there is a great deal of important Babylonian mathematical material in London, Berlin and elsewhere, its published interpretations hold extraordinary promise for the future enrichment of our knowledge of mathematics in these early days.

Until recently we have connected the Babylonians more with astronomical than with mathematical achievement, for they were famed for their knowledge of the heavens, and the pseudoscience of astrology originated with them. They were acquainted with the

[72] B. A. Turaev, *Ancient Egypt*, 1917, p. 100–102.

[73] Gunn and Peet, *Journal of Egyptian Archaeology*, 15: p. 176.

[74] *Journal of Egyptian Archaeology*, 15: 179–184.

[75] F. Röber, "Beiträge zur Erforschung der geometrischen Grundformen in den alten Tempeln Aegyptens und deren Beziehung zur alten Naturerkenntniss," Dresden, 1854, p. 15–16.

F. Röber, "Elementar-Beiträge zur Bestimmung des Naturgesetzes der Gestaltung und des Widerstandes, und Anwendung dieser Beiträge auf Natur und alte Kunstgestaltung," Leipzig, 1861, p. 20–22.

[76] *Philosophical Magazine*, s. 4, 27: 124–132, 1864; R. P. Graves, "Life of Sir William Rowan Hamilton," vol. 3, Dublin, 1889, p. 141–148, 581–587. For other details in this connection see my Bibliography under 1854, and my notes in *American Mathematical Monthly*, 28: 477–479, 1921.

[77] I have referred in my Bibliography to a large unpublished geometrical Greek papyrus in Vienna found in the Fayûm and dating from about 350 B. C.

five planets Mercury, Venus, Mars, Jupiter and Saturn, distinguished the signs of the zodiac and made long lists of the fixed stars. The chief duty of the astronomer was to observe the moon for the purpose of keeping the calendar.[78] Their predictions of eclipses were made by a rule based upon the empirical discovery that there was a period of a little over eighteen years within which eclipses repeat themselves.[79]

That Greek astronomy was based on the astronomy of the Babylonians is well known. We have noted that certain mathematical results regarded as original with the Greeks must, from now on, be attributed to the Babylonians and to the Egyptians. But whatever new facts are found concerning mathematics before the Greeks, the glory of their achievements in creating a vast body of deductive geometrical reasoning is not likely ever to be dimmed.

[78] British Museum, "Guide to the Babylonian and Assyrian Antiquities," London, 1922, p. 25.

[79] For a summary of Egyptian astronomy see T. E. Peet, "The Sciences" in "The Cambridge Ancient History," Cambridge, vol. 2, 1924, pp. 218, 656.

OBITUARY

EDWARD DRAKE ROE, JR.

DR. EDWARD DRAKE ROE, JR., for twenty-nine years professor of mathematics at Syracuse University, died suddenly at his home in Syracuse on Wednesday, December 11.

Dr. Roe had a long and distinguished career as student, educator, mathematician and astronomer. He received his bachelor's degree from Syracuse University in 1880. He then went to Harvard University where he first studied medicine and then returned to his studies in mathematics. He received a bachelor's degree from Harvard in 1885 and a master's degree in 1886. After teaching a few years, he went to the University of Erlangen, Bavaria, where he won his doctorate in 1898. He returned to America and was made associate professor of mathematics at Syracuse University.

In 1901 he was awarded the John Raymond French chair of mathematics and some years later was appointed director of the Holden Observatory of Syracuse University.

Dr. Roe built his own private observatory in connection with his own house. It was equipped with an Alvan Clark equatorial telescope and is considered one of the best-equipped private observatories in the country.

He was the author of nearly seventy scientific articles on mathematics, astronomy and philosophy. He was the author of a text-book in trigonometry and one in algebra.

He was the founder and director general of the honorary mathematical fraternity, Pi Mu Epsilon, and was a member of the Phi Beta Kappa, Sigma Xi, Pi Kappa Phi and Delta Kappa Epsilon. He was a fellow of the American Association for the Advancement of Science, a member of the American Mathematical Society, the founder and president of the Syracuse Astronomical Society, a member of the Deutsche Mathematiker Vereinigung, Circolo Matematico di Palermo and Société Astronomique de France.

Dr. Roe stood for high scholarship, thorough scientific study and research. He impressed all who knew him as a scholar with a deep thirst for knowledge. He worked with untiring patience in mathematics and its allied science astronomy in the university and in the community.

Throughout his years of service as a teacher Dr. Roe always stood for the highest ideals of intellectual honesty and scientific achievement. He was a devoted teacher, a deep thinker, a philosopher and an earnest Christian. In his death Syracuse University has suffered a distinct loss.

ALAN D. CAMPBELL

MEMORIALS

THE American Electrochemical Society has announced its intention of establishing the Joseph W. Richards Memorial Fund, the interest of which is to be used as an honorarium to foreign electro- and physical chemists who are to be invited by the society from time to time. The guests will present lectures at annual spring conventions of the American Electrochemical Society, and possibly at universities and other institutes of learning. Professor Richards was secretary of the American Electrochemical Society for almost twenty years and was very largely responsible for the founding of the society. He was ever a very strong advocate in fostering better relations between our own scientists and scientists abroad, inviting and entertaining many notables at his own expense. The many friends of Professor Richards, therefore, feel that this memorial to him is a most fitting one. All those interested are invited to send their contributions to the secretary of the American Electrochemical Society, Columbia University, New York City, making all checks payable to the Joseph W. Richards Memorial Fund. It is the desire of the Board of Directors of the society that the list of contributors include as many as possible of Professor Richards'