resigned as associate professor of physics to accept an appointment at the Smithsonian Institution. Dr. Francis A. Jenkins, of New York University, has accepted a position as assistant professor of physics. As has already been noted here, Dr. Leonard B. Loeb has been promoted from an associate professorship to a professorship of physics.

DR. CLIFF STRUTHERS HAMILTON, associate professor of chemistry at Northwestern University, has been appointed to a professorship in the department of chemistry of the University of Nebraska.

DR. D. G. STEELE, of Yale University, has been appointed assistant professor of genetics in the resident teaching division and assistant geneticist in the experiment station division of the Connecticut Agricultural College.

DR. CLYDE A. MALOTT, of Indiana University, has been appointed professor of geology at Williams College.

DR. JULIA M. SHIPMAN, of the University of Tennessee, and Professor Leland S. Paine, of the Texas Agricultural College, are visiting lecturers who will give courses in geography in the summer session of the University of Nebraska, which opens on June 10 and lasts nine weeks.

DR. JAN SCHILT, research assistant at the Yale Observatory, has been promoted to be an assistant professor.

DISCUSSION

REMARKS ON UNCERTAINTY PRINCIPLES

SINCE the publication of Heisenberg's paper[1] on the "anschaulichen Inhalt" of quantum mechanics, discussions of the fundamental limitations on the accuracy of physical measurements have been much in the foreground. According to Heisenberg, the quantum mechanics implies that it is impossible to measure simultaneous values of a coordinate and its conjugate momentum with unlimited precision. Instead, if Δp be the estimated error or uncertainty in a momentum and Δq that in the associated coordinate one must have the inequality,

$$\Delta p \Delta q > \frac{h}{2\pi} \qquad (1)$$

This inequality has come to be known quite generally as Heisenberg's uncertainty relation.

In discussions on this subject it is essential to distinguish two standpoints. One is the analysis of proposed experiments, whether realizable or ideal, by which it is proposed to make measurements. The other is that of the relation of the uncertainty principles to the laws of quantum mechanics as now formulated. It is only the second standpoint which is considered here.

The origin of the uncertainty relation (1) for p and q lies in the fact that the operators which represent p and q do not commute. Therefore one is tempted to suppose that such an uncertainty relation may be true for any two quantities whose operators do not commute. Such, however, is not the case. These remarks establish by means of specific examples the truth of the following statements:

(*a*) The fact that the operators corresponding to two physical quantities, A and B, do not commute does not imply the existence of an uncertainty relation of the form of (1), namely, that the product of the two uncertainties must be greater than or equal to some lower limit.

(*b*) Even if A and B do not commute, there may be exceptional values of A and B which may be both known simultaneously with no uncertainty.

(*c*) There may exist a limited class of states of the system, with regard to which A and B do commute, but in which nevertheless the two quantities A and B can not be known with unlimited precision.

The relation of the uncertainty principle to the quantum mechanics may be formulated as follows. The configuration of a dynamical system of n degrees of freedom is specified by n spatial coordinates, as in classical mechanics. There may appear new coordinates which do not have classical analogs like the electron spin or the permutation variables but these will be left out of account. The particular state of the dynamical system at any instant is then specified by giving a function $\varphi(x_1 \cdots x_n, t)$ which has the property that $\bar{\varphi}\varphi d\tau$, where $d\tau$ is the volume element of the configuration space, is the probability that the system be found at the instant, t, with its configuration lying in the volume element $d\tau$ of the configuration space which surrounds the point, $x_1 \cdots x_n$.

Corresponding to each physical quantity there is a linear operator, which when applied to φ gives another function of the coordinates, $x_1 \cdots x_n$. Let A be a physical quantity and at the same time A may stand for the operator which represents A. Then

$$\int \bar{\varphi} A \varphi d\tau$$

represents the mean or expected value of A^2 in this state of the system characterized by φ. The integration is over the entire configuration space. Similarly,

$$\int \bar{\varphi} A^2 \varphi d\tau$$

represents the mean or expected value of A^2 in this state of the system.

[1] W. Heisenberg, *Zeits. für Physik*, 43: 172. 1927.

We shall define the uncertainty in the value of A associated with this state by the equation,

$$(\Delta A)^2 = \int \bar{\varphi} A^2 \varphi d\tau - \left(\int \bar{\varphi} A \varphi d\tau \right)^2 \qquad (2)$$

where ΔA is written for the uncertainty in A. This evidently corresponds with the classical definition of the uncertainty as the square root of the mean of the square of the deviation from the mean. One observes that if φ is such that $A\varphi = a\varphi$, where a is an ordinary arithmetical number, then ΔA vanishes. In such a case the value of A is precisely a. For example, if A is really the Hamiltonian function and φ is one of the solutions of Schrödinger's equation for the system, one has the proof that the Schrödinger wave functions correspond to states of the system in which the total energy has precisely one of the allowed values of the energy.[2]

The examples illustrating propositions (*a*), (*b*) and (*c*) are afforded by considering the different components of angular momentum of a particle about the origin. These will be denoted by M_x, M_y and M_z. As is well known, the operators corresponding to these three quantities do not commute with each other, the operators being,

$$M_x = \frac{h}{2\pi i}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \qquad (3)$$

the other two being given by cyclic permutation of x, y and z. These operators satisfy the following commutation rules,

$$M_x M_y - M_y M_x = \frac{h}{2\pi i} M_z, \qquad (4)$$

and two others given by cyclic permutation of x, y and z.

As to (*a*), we observe that the operator for M^2, where

$$M^2 = M^2_x + M^2_y + M^2_z,$$

commutes with the operator for any component, M_x, M_y or M_z. If we consider in particular states where φ is of the form

$$\varphi = R(r) e^{im\phi} P^m_l (\cos\theta), \qquad (5)$$

where R(r) is any function of r, and the z axis is the pole of the spherical polar coordinate system, it may be readily verified that such states correspond to precise values for M^2 and M_z given by

$$M^2 = l(l+1)\left(\frac{h}{2\pi}\right)^2 \text{ and } M_z = m\frac{h}{2\pi}$$

If one compute the value of ΔM_x or ΔM_y for such states it turns out to be equal to

$$(\Delta M_x)^2 = (\Delta M_y)^2 = \tfrac{1}{2}\,[l(l+1) - m^2]. \qquad (6)$$

[2] This formulation corresponds to that of Weyl, "Gruppentheorie und Quantenmechanik," Leipzig, 1928, p. 67.

This is finite for finite values of l and m. Therefore we have here a class of states of the particle in which, since ΔM_z is zero, the product of the uncertainties in ΔM_x and ΔM_z is zero, in spite of the fact that the operators M_x and M_z do not commute. Hence the truth of (*a*).

As to (*b*), we notice that if

$$\varphi = R(r) \qquad \text{(independent of } \theta \text{ and } \varphi)$$

this corresponds to the precise values,

$$M_x = 0, \quad M_y = 0, \quad M_z = 0,$$

with no uncertainty in any of them. Hence the truth of (*b*).

As to (*c*), we observe that if we deal with the class of states in which M_z is known to have the value zero precisely, then from the commutation rule (4), applied to any φ of this class, the operator $M_x M_y$ gives the same result as $M_y M_x$. But nevertheless the uncertainty in neither M_x nor M_y is zero in such a state. The φ functions for this class of states are evidently of the type of (5) with m set equal to zero, so the preceding calculation of ΔM_x and ΔM_y given in (6) applies here. Hence the truth of (*c*).

It would appear, therefore, that a general uncertainty principle is not simply to be formulated in terms of the commutativity or lack of commutativity of the operators associated with A and B, as is usually implied in discussions on this subject. What the exact criteria may be, we are not prepared to state.

Although largely of a negative character, these remarks have considerably clarified the situation for me. In working them out, I have derived much benefit from stimulating conversations with my colleagues, especially Drs. J. E. Mack, E. C. G. Stueckelberg, G. P. Harnwell and H. P. Robertson.

E. U. CONDON

PALMER PHYSICAL LABORATORY,
PRINCETON, N. J.,
MAY 10, 1929

HONORARY DEGREES AND A SUGGESTED OPPORTUNITY

BEFORE many moons in American universities will come the harvesting time of the four years' crop of A.B.'s, B.S.'s and other varieties of B's less well known. At the same time will come to maturity the smaller crop of Ph.D.'s and D.S.'s which have had a somewhat longer ripening period under the watchful care of the academic gardeners. These exhibits are deservedly of much local interest, but it is the exceptional specimens, the men with honorary degrees, who as the big pumpkins of the harvest festival attract most attention and gain the front page of our newspapers. It may not be amiss, in the interest of