certain types of cells, as though by a surgical operation of surpassing delicacy. We can also reach within the cell and effect changes, particularly in the nucleus. It seems that we can even change the genes and thus inheritance. Most important of all, irradiation promises clues to basic physiological processes. In medicine it has found many applications. It may assume equal importance in the breeding of domesticated plants and animals. In the general field of biological science it offers a new technique before which old problems may fall.

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SPACES OF STATISTICS AND THEIR METRIZATION[1]

WE may generalize the ordinary statistical graph by considering an n-dimensional space in which each coordinate represents a statistical variate. How much arbitrariness is there in the choice of a particular coordinate system? In many familiar cases almost any change of coordinates is meaningless; geometry can then hope to throw but little light on the problem, compared to that which it supplies for physical problems, in which such transformations as those of the Euclidean group are held not to change the nature of the case. But certain situations arise in which real invariant properties exist for rather extended groups of transformations.

One of these cases is in economics. Let $p_1, p_2, \ldots, p_n$ be the prices of n commodities; and with these prices ruling, let $q_1, q_2, \ldots, q_n$ be the respective quantities that can be sold. Now the same market situation can be expressed otherwise by giving the commodities different definitions; *e.g.,* a certain quantity of iron and wood will make a certain number of Fords and a certain number of garages; if we know the equations of transformation we have the same information given by either pair of quantities. Wheat of different grades can be, and is, mixed in different ways to meet the grading systems in different countries. We can readily prove that under these transformations in the manifold of prices, quantity is a covariant vector. Dually, in the manifold of quantities price is a covariant vector.

Another example is in biology. If an individual be regarded as fully specified by the dimensions of n organs he is, by definition, a point in a space of n dimensions. A species is a cluster of points, and may be typified by its center of gravity. Coordinates may be transformed by changing the methods of measurement; thus we may give the height of a man's shoulder and the length of his arm, or we may give the height of his hanging hand and the length of his arm.

But in all this no invariant distance element has appeared. We are impelled to look about for some quadratic form, preferably a quadratic differential form, of statistical significance.

Now a quadratic form of overshadowing importance is found in the exponent of the normal law of probability in n variables. The various reasons for adopting this law are strongest when the deviations of which the quadratic form is a function are small, and so it is natural to take it as a quadratic differential form.

The idea of random migration has received much mathematical consideration by Karl Pearson, Lord Rayleigh and others. It may be applied to particles in a biological n-space to discuss evolution. A son differs from his father in n ways; if these deviations are all independent and have equal dispersions, the probability of a set of deviations $\delta x_1, \ldots, \delta x_n$ is proportional to e^{-T}, where T is a constant multiple of $\delta x_1^2 + \delta x_2^2 + \ldots + \delta x_n^2$, and so may be said to define the *distance* from father to son. If the deviations are not independent, product terms $\delta x_i \delta x_j$ appear in T. If finally the dispersions and degrees of interdependence of the δx's depend on the x's we have the quadratic differential form with variable coefficients, and therefore a Riemannian geometry. That the dispersions do in fact vary with the size is evident by considering the difference in centimeters in the length of a pair of twin elephants, and then comparing this with the variability in length in a litter of mice. It may safely be assumed that the intercorrelations, for a given species, may also vary with size and shape.

In this way we obtain a metrical space, in general curved, as a matrix for possible organisms. A species, represented by a swarm of particles, diffuses gradually by an accumulation of small changes in a manner analogous to the conduction of heat in this curved space. It would eventually become so diversified as to supply the naturalist with every conceivable kind of specimen were it not for the effect of selection. This may be pictured as a system of heat "sinks," of refrigerated localities, spread here and there to trap and annihilate unwary individuals. On the whole the losses due to the sinks are of course made good by natural increase.

The center of gravity of a swarm of individuals may be taken to represent the species. Suppose that we have a collection of fossils which shows us the location of this point at each of two times a million years apart; it is desired to know its most probable positions at intermediate times. If we have no

[1] Presented to the American Mathematical Society, September 9, 1927.

"sinks," no selective action, then the most likely path is a geodesic. If we do have "sinks" they act as propulsive forces and we encounter the following generalization of the theory of differential equations. At each point, instead of having a definite direction of motion assigned, as by differential equations, we have a pencil of directions (or an (n–1)-parameter family of directions) with a function, which will be a normal error function, giving the probability that the direction of motion will lie between any assigned limits. This enables us to assign to the curves joining two points a distribution of a species of probability. For fixed end points the selection of the most probable curve is a problem in the calculus of variations. If g_{ij} is the fundamental tensor and φ_i a vector giving the most probable direction we shall in fact minimize

$$\int_{t_1}^{t_2} \sum_{i, j=1}^{n} g_{ij}(\dot{x}_i - \varphi_i)(\dot{x}_j - \varphi_j)\, dt,$$

where $\dot{x}_i = dx_i/dt$.

The minimizing equations for the integral may be interpreted as the differential equations of a dynamical system. Indeed we may consider the most probable path as the trajectory of a particle shooting through the curved space under a field of force. However, the form of the integral shows that the force will in general depend upon the direction of motion, as with an electrical charge moving in a magnetic field.

The derivation of the criterion that the integral shall be minimized is a simple generalization of a derivation in the article on "Differential Equations subject to Error, and Population Estimates,"[2] in which I use the same considerations in their simplest form to obtain estimates of intercensal populations. That was in one dimension, but it may readily be generalized by considering a system of variables having correlated changes, such as the population of a city, the number of children in its schools, the number of telephones and so on. The leading difficulty in applications of this kind is to arrive at the tensor g_{ij}.

There is nothing to prevent empirical determination from measurements of parents and offspring of the fundamental biological tensor giving the distance element which we have defined. In fact all the measurements which have been made in the study of heredity may be regarded as steps in this enterprise. As a sufficient accumulation of data makes the hypothesis of flatness untenable we shall be driven to look for some other kind of space, as simple as is possible without contradicting the data. A space which in some sense or other has constant curvature will be wanted as a second approximation, and mathematicians will be asked to supply equations of suitable type, with parameters for biological workers to determine empirically and check by tests of goodness of fit.

This problem which biology is approaching has already been faced by physics. The hypothesis of Euclidean space-time as a matrix of physical events served adequately for several centuries; but more refined measurements have required its modification. Mathematical physics has been grappling with the problem of supplying as simple a statement as possible of the properties of the world-order without contradicting known facts. Such a statement is indeed what we call an explanation. It is altogether likely that the considerations of simplicity which led Einstein to his cosmological equations may some day cause the same equations to appear as the foundation of biology.

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2 *Journal of the American Statistical Association*, September, 1927, pp. 283–314.

PAUL HEINRICH VON GROTH

It is with deep regret that we record the death of Professor Paul Heinrich von Groth in Munich, Germany, on December 2, 1927. With the passing of Professor Groth the Mineralogical Society of America has lost one of its most distinguished honorary life fellows, the science of mineralogy one of its greatest leaders and the world of science a courageous pioneer, an ardent investigator, an energetic and efficient author and editor and an inspiring teacher.

Paul Heinrich von Groth was born on June 23, 1843, at Magdeburg, Germany. His father was a portrait painter. The training for his life's work Professor Groth obtained at the school of mines in Freiberg, at the college of engineering in Dresden and at the University of Berlin, at which institutions he spent the years 1862 to 1870. The degree of doctor of philosophy was conferred upon him by the University of Berlin in 1868.

From 1870 to 1872 Professor Groth was a member of the teaching staffs of the Technische Hochschule in Charlottenburg and of the University of Berlin. When the University of Strassburg was being reorganized, shortly after the close of the Franco-Prussian war, Groth was called to the chair of mineralogy, for he had already acquired a splendid reputation as an investigator of great promise, especially in the field of chemical crystallography, to the development of which he subsequently contributed so extensively.

Groth held the professorship at Strassburg from