tional viewpoints, as exhibited in the treatment of medicine in the universities, there is the question that is bound to be raised as to whether the scientific method is, after all, the most effective one for the advancement of knowledge in this particular realm. Great advances have undoubtedly been made in this field through a purely empirical approach. The Nobel prize has just been awarded for a discovery in this field which was not made through what is considered the scientific method. This association believes, however, that the best approach to knowledge is through the gateway of science.

The science of medicine requires workers imbued with the scientific spirit. Opportunities must be available for men who want to learn about disease. These men must be stimulated, and their work given appreciative recognition by workers in better established fields. The bestowal of this recognition and stimulation offers one way in which science may be advanced by this fellowship of scientists.

In what has been said I have merely tried to give expression to what has already been in the minds of most of us. The programs of recent years, and the program of to-day, indicate that the motivating force in this society is the promotion of scientific medicine. Through the efforts of this organization may medicine ever become more scientific, to the great blessing of mankind.

RUFUS COLE

THE ROCKEFELLER INSTITUTE
FOR MEDICAL RESEARCH

TOO LITTLE MATHEMATICS—AND TOO MUCH[1]

IN speaking on a subject like mine of to-night it is necessary to proceed by classification, for what might be too little mathematics for some groups of students could well be too much for others.

First let us consider those who contemplate becoming professional mathematicians. These form a negligible group—negligible in numbers compared with all those who should take some collegiate mathematics, negligible for purposes of instruction because each will go more or less his own precocious way. It is improper for any department selfishly to lay its plan of teaching for the rare Galois or Gauss or Abel, or even for the commoner Richardson or Archibald or Wilson. The future professional who is worth his salt will get along somehow without much of our assistance and possibly even in spite of it. He will follow the courses he wants and make up his deficiencies by private study. The professional knows how to learn, and in mathematics he knows it young. There will be small chance that he have too little mathematics; you may think he can hardly have too much. Herein I should differ with you and maintain that his real danger is "too much." I do not, of course, refer to what he will have learned or may have forgotten by the time he comes to a ripe old age or even by the time he reaches middle life; I am speaking of the future professional mathematician as a student, first collegiate and then graduate. He should not be stuffed with courses until his brain has become a glorified paté de fois gras. That would deaden his originality, doubly diminish his ingenuity, and thus triply hinder his development as a true professional.

Our American university departments sin greatly in the great array of prerequisite specialized courses which they offer and expect their students to take. The aim of any advanced course should be to line out a straight path from relatively elementary work to the research line. When I was in Paris I took a course at the Sorbonne on the dynamical theory of light with Boussinesq. He did not hesitate to take his time to explain that $\cos(\pi - x)$ equals $-\cos x$ or to prove some ordinary proposition in solid analytical geometry or to integrate some common differential equation, and yet in 60 lectures he reached material to be found only in his own recent papers. When I asked him why he took the time to prove so many elementary theorems he said that it did the students good to see such propositions demonstrated and that he had a superabundance of time anyhow. Such a course in any one of many fields could be given in our American colleges to seniors who had had three consecutive years of mathematics, if only our teachers had the finesse to give it. A book must be somewhat encyclopedic; a course should be selective, eclectic, for the purpose of helping the student quickly to an original problem upon which he may go to work. You have here at Brown one of the best collegiate departments of mathematics in the United States; you have an able staff; you need not hesitate to offer the doctorate. What you may miss is encyclopedicity, you will for that very reason the more easily make up in the freshness and promptness with which you start on original investigation.

Let us turn from the future professional to the ordinary college man. He may well have too little mathematics, too little for his own future comfort. The college student of inorganic chemistry of 30 years ago who failed to take a good course in calculus has paid heavily for this omission. It has much increased his difficulties in physical chemistry, in the work of van't Hoff and of Nernst, of Lewis

[1] Read before the Mathematical Club of Brown University.

and of Debye. For even longer any one who expected to go on into physics, astronomy or engineering should have had his calculus. To-day the physiologist, the bacteriologist, the biologist, and the student of the public health is reaching problems which require some mathematical insight for their elucidation, and the college student of to-day can scarcely fail to find himself seriously handicapped before middle life if he enters upon any of these fields without a college course in calculus. For many, many years not only arithmetic but algebra and geometry have been regarded as so fundamental in all education both for their own sake and for their uses in other studies that they have been required for entrance to college and have been widely taught even to those who did not plan to go to college. It may to-day safely be said that calculus is for the college student what algebra and geometry are for the high school pupil—of necessity for future use in many a profession, of cultural value for many a student who does not contemplate a profession. So much can not be said of trigonometry—biology, chemistry, physiology, and those branches of business, of medicine and of the public health which depend on statistics hardly use a periodic function; it is the logarithm and exponential which are central. There is small use also for analytic geometry. Graphs we do find everywhere, but the art of deducing theorems by algebraic processes is for the most part special to pure mathematics.

There is real danger that the college student shall have too little mathematics. To-day more are taking too little than too much. Why this serious situation? We may find some light on it if we reflect on the case of the chemist now in middle life. Most likely he took a full year of mathematics in college. Thirty years ago in the great majority of our colleges he had no option but to take it. However he did not learn calculus unless he followed mathematics for two or even three years. In other words we may say that between his algebra of the preparatory school and the method of dealing with variable quantities that he needed there was intercalated by the habits of the mathematical teachers of those days one or two years of work which could at best be of very limited use to him. Conditions are better to-day. We generally get to some calculus in the freshman course. I feel, however, that our collegiate teachers of mathematics could well go somewhat further than they yet have gone in making straight and fair the path of the ordinary student to and through the elements of the calculus both differential and integral.

Now there are two aspects in almost all learning; there is the logical or intellectual side which deals with the meaning of a subject, which is chiefly emphasized in elementary geometry, and the technical or operational side, which is largely placed to the fore in algebra. You may be a first class technician without much understanding of the significance of a subject or you may have a fast intellectual grasp with a negligible technique; you may be a thinker or a doer. These two activities are not entirely antagonistic, to a considerable extent they reinforce each other. In far the greater part of our life we are technicians of one sort or another, we learn how to do certain things. We do not go about learning to walk by first studying our skeletal structure, our musculature, and the laws of the lever. Whenever we need to do something repeatedly with precision we face the problem of developing a technique. The billiard player does not spend his time studying Routh's Rigid Dynamics. He acquires a sort of automatism under voluntary control. We can do the same for factoring or for differentiating. But what is the use; the great majority of us do not take naturally to such pursuits any more than we do to billiards; we have little occasion to use formal mathematics except of the simplest.

It seems to me that one error in much teaching of mathematics for the majority of students is overemphasis on technical processes and correlative underattention to meanings. It is too much mathematics of one sort and too little of another. According to my experience it is very difficult to expect a student to learn good technique and to gain a clear understanding at the same time. I prefer to aim the first general collegiate course at the understanding and to follow this up for those who are to be engineers or mathematicians, physicists or astronomers, and for those who like mathematical operations for their own sake, by a stiff course in technique. Now if you are going to teach or to learn meanings of things, you must not be in a hurry, you must keep your self-possession of mind, you must dwell long on a variety of simple things. What is the problem of learning the calculus so that you may use its ideas in any one of a great variety of applications which may later confront you as chemist or physiologist, as economist or public health officer? You have to get a real understanding of rates and summations, of infinitesimals and infinites, of limits of quotients of two infinitesimals and of sums of an infinite number of infinitesimals. That is about all; it is not much, but it takes time and patience. It requires the avoidance of early generalization. Mathematicians have a fondness for general methods in instruction. They forget that in their own work they almost always use some ingenious special notion and rarely the most general method they know. Not many of us naturally work with E. H. Moore's "General Analysis"! In teaching, one must keep the special to the fore and let the generalization come later as a union in a single view of a variety of specials already known. And in general, one must proceed from the known to the unknown. That is why review is constantly necessary, why the repetition of a demonstration from previous studies is not a waste of time, but actually a time-saver. We do not ourselves readily use that last thing we have learned any more than we use the most general; why expect of our students that which we have learned by experience we may not expect of ourselves? The breathlessness of American life is an index of indigestion. We are stuffed with the latest news, the most recent invention, the newest fad whether of mechanism or of dancing, of science or of sociology. We do not know what they mean, we do not even inquire what are their antecedents from which an inkling of their meaning might be learned. The classics have pretty much vanished from our instruction and science has not yet learned to replace them in poise and in perspective.

Perhaps I might without boring you go somewhat into detail about one way of introducing the few fundamental concepts along which those who need a little mathematical thinking may be led without too much involvement in formal operations. Begin with uniform velocity and its space-time graph of which that velocity is the slope. This is a better definition of slope than through the trigonometric tangent. Follow with uniform acceleration, treating variable velocity in the case of uniform variation as the innate concept that is. Point out the logical and geometrical identity between the relations of uniform acceleration to velocity and time and the relations of uniform velocity to space and time. As much of our fastest thinking is by analogy, it is well to emphasize true analogies. Proceed to deduce by any one of the old fashioned methods the relation between distance and time in uniformly accelerated motion. Then deduce the relation over again by a variation of the argument. Two proofs of one important central theorem which involves the most fundamental of notions give more power and understanding to the student than the proofs of two theorems of which only one is important. Dally over the properties of the quadratic or parabolic space time graph, $s = \frac{1}{2} at^2 + bt$, not so much for the importance of those properties in themselves as for their significance to uniformly accelerated motion. It is remarkable how simply some of the geometry of the parabola develops this way. The fact that any chord of a parabola is parallel to the tangent at the abscissa half way between those of the ends of the chord is but another statement of the proposition that in uniformly accelerated motion the velocity at the mid time of any interval is the mean velocity. That the tangents at the ends of a chord meet on this same mid abscissa means merely that you will cover the same distance whether you move with uniform acceleration or with the initial velocity for half the time and then with the final velocity for the remaining half. So too the fact that the line joining the intersection of two tangents to the mid point of their chord of contact is bisected by the parabola is simple kinematics. All this is helpful for reinforcing the meaning of things fundamental.

When you have reached this point you can hardly resist the temptation of solving the quadrature of the parabola by proving that the curve divides the triangle formed by two tangents and their chord of contact is the ratio 1 : 2. It is a mere corollary obtained by doubling up the construction, and so much simpler than the quadrature of the circle. There is no infinite series to sum; the construction at each step cuts off half as much on the outside as on the inside. And then you have Simpson's Rule. You can take your students back to the quadrature of the circle just for review and to show them how much harder it is. Moreover you have the indefinite quadrature of the parabola, whereas for the indefinite quadrature of the circle you need not merely π but the inverse trigonometric functions. However, you may be strong minded and able to resist this little excursion into quadrature. You could return to uniform velocity to discuss interpolation by proportional parts and then with the aid of uniform acceleration pass on to interpolation to second differences. This latter subject is often, perhaps generally, omitted from mathematical instruction. I have run across more than one candidate for the doctorate who did not know that the condition for safe interpolation with first differences is that the second differences are less than 4. It is true that second differences are not important for themselves; they are, I think, important for the insight they give into the behavior of infinitesimals, and to my mind the infinitesimal is the fundamental thing in calculus. I hold no brief against limits. They are necessary to clear thinking, including clear thinking about infinitesimals. I will not further go into detail, but merely point out that if the teacher has done a good job on uniform and on uniformly accelerated motion and on first and second differences, he is ready for definitions of limits, of infinitesimals, of derivatives and of integrals with the assurance that he has provided kinematical and arithmetic considerations on which those definitions may be cogently illustrated and subsequent developments may be safely built.[2]

My thesis is that if we are to encourage the general student, who ought to be encouraged, to take a little mathematics we should avoid subjects which will be of small use to him and in particular should emphasize ideas and simple applications of the calculus rather than inflict upon him from the start a vast operational technique. We should use simple concrete ingenious special methods instead of setting up general systems of analysis. In short we should consider the student somewhat in the light of our own daily experience properly written down to his present level. We must avoid giving him too much mathematics if we are fairly to expect him to avoid taking too little. According to my experience it is rare in departments of mathematics to hear such questions discussed. Indeed I sometimes think that around universities one seldom hears a discussion of any matter of education, of teaching; there is a lot about administration and about research, both of which I should suppose existed chiefly in the college for their reactions on teaching rather than for themselves. It may be that pedagogic questions are so far settled as no longer to need original and critical thought; at any rate, there is in the vast supply of text books produced by college professors much more of uniformity than in their research, and the uniformity is of a type which suggests a study of other recent and similar texts more than of the history of science, of the method of approach to new subjects, of the great texts of all time.

We have considered together the possibility of too little or of too much mathematics for the future professional and for the general student. Through the exigencies of the institutions in which I have worked, that is through opportunities which have arisen and which it has seemed desirable that I should meet, my course as a teacher has laid successively through modern geometry, mathematical physics, mechanics, aeronautics, physics and now rests in vital statistics. Statistics is peculiarly a subject in which too little or too much mathematics may be used and in which a golden mean is especially necessary to soundness of judgment. When I was in college one might find some astronomer teaching least squares as a method of reducing observations, some physicist lecturing on kinetic theory, some mathematician expounding probability; but statistics was practically unknown to the most varied curriculum. Indeed Yule's text appeared in its first edition only in the year 1899 of my graduation from Harvard. Now everything is changed. There are many courses undergraduate and professional on statistics and a new book comes out almost every other month. What is the intellectual level of these courses and these books? For the most part, so far as I can determine, that level is of the lowest, not comparable with pre-college mathematics or classics, about on a par with manual training. There are presented arrays of tabular material, of methods of calculation and of computation forms, without emphasis on logical analysis, still best represented by the 25-year-old text of Yule, without any substantial introduction to probability and chance, which except for logic gives the only firm foundation upon which to build an understanding of the difficult and treacherous science of statistics. It is as though you should teach applied mechanics without going into mechanics or physics or calculus. It is an exaggerated case of too little mathematics. Of course if our colleges and universities wish to train simple technicians in statistical methods who shall work their arithmometers under the direction of competent statisticians much as draftsmen work under designers, this absence of mathematics and of intellectual attention in statistics may be advantageous. But I take it that no considerable part of the students who follow statistics will be technicians, that each statistical problem which they may meet in after life will require its own formulation prior to any application of technique and will ordinarily be solved, insofar as it is solved, by an exercise of judgment rather than of technique. There is little excuse for not giving a sound theoretical background to statistics; the necessary mathematical technique is minimal. As is pointed out by Fréchet and Hallwachs in their excellent little book, "Probabilité à la Portée de Tous" (probability for everybody), arithmetic and algebra plus a willingness to think hard on knotty points are the requisites; the use of calculus though helpful fairly early can be postponed if unfamiliar to a late stage that is seldom reached in a first course in statistics.

We sometimes see in the press a pronouncement by Nicholas Murray Butler on politics, and again on prohibition. Presumably it was on one of these latter occasions that some scalawag of a compositor is said to have headlined him as Nicholas Murray, butler of Columbia University. What I fear many of us do not see or know are his equally strong trenchant dicta on education. His annual presidential reports are interesting reading and repay study. Let me quote a few detached sentences from the latest:

> Through ignorance the present-day banners of progress are everywhere emblazoned with the names of some of the oldest of humanity's discarded failures. . . . Nor is it in any wise true that all subjects of intellectual interest are of equal value and that the important thing is not what one studies but how he studies it. . . . Just now there is a strong tendency to exalt unduly certain recently devel-

[2] This type of course on the introduction to the calculus could be given in the first semester of the freshman year in forty-five exercises.

oped fields of knowledge which as yet consist almost entirely of futile talk and unproved opinion. . . . The longer one examines programs of study that are now most widely followed, observes the spirit in which school and college teaching is so often carried on, and notes the careful avoidance of anything that makes for genuine scholarship and power of reflective thinking, one is forced to raise the very far-reaching question, whether we have not destroyed the ideal of the liberally educated man and, with it, the liberally educated man himself.

I could quote more, but this is perhaps enough amply to cover what I said earlier about dwelling on the ideas of mathematics in general instruction and what I have just said about the lamentably low intellectual level of most of our courses and texts on statistics. I do not care to have any student making routine arithmetical calculations unless he is mentally equipped to understand the limitations of such methods and the conditions under which such calculations lead to legitimate inferences. Some training of fingers is a necessary accompaniment to the elevating of a brain, but the brain is the important thing for those who would be more than technicians.

Although too little mathematics keeps one from understanding statistics, too much is almost as bad. The mathematician who teaches this or any other applied science is very likely to err on the side of too much. It is a great mistake to think that because one finds his original material in the field of statistics or physics or engineering or physiology, subsequent mathematical analysis must be a contribution to that field—it may be just as pure mathematics as if it had originated as such. Much mathematical statistics has no significant statistical content. Many a memoir on mathematical physics is vacuous of physics. Once late in a course on mathematical physics Willard Gibbs came to set up and discuss the equations of motion of the top. In the course of the work he turned around to us, his face lighting up with a sweet smile, and remarked: There are some who seem to think that the top is chiefly interesting as furnishing exercises in elliptic functions. One branch of statistics where mathematics may easily run wild is in fitting frequency functions. Consider the tables printed in the adjacent column which give the frequency distribution of infant mortality rates for 1918 in I. Cities over 25,000, II. Rural Counties for white infants and III. Rural Counties for colored infants. The data are grouped for convenience. Clearly the rate might within the limits of its range have any value and although you can not have a continuous distribution of a finite number of elements you have here finite samples chosen from theoretical infinite universes and are confronted with the problem of fitting to these data some continuous curves which shall represent as well as may be the dis-

INFANT MORTALITY RATES IN 1918

I. CITIES OVER 25,000

Rate	Frequency
0– 19	0
20– 39	0
40– 59	1
60– 79	17
80– 99	43
100–119	40
120–139	27
140–159	13
160–179	1
180–199	2
200–219	0
220–239	0
	144

II. RURAL COUNTIES, WHITE

Rate	Frequency
0– 19	0
20– 39	9
40– 59	33
60– 79	63
80– 99	64
100–119	38
120–139	16
140–159	8
160–179	2
180–199	0
200–219	1
220–239	0
	234

III. RURAL COUNTIES, COLORED

Rate	Frequency
0– 19	1
20– 39	1
40– 59	5
60– 79	22
80– 99	19
100–119	42
120–139	37
140–159	30
160–179	19
180–199	17
200–219	10
220–239	12
240–259	8
260–279	4
280–299	2
300–319	1
320–339	0
340–359	0
360–379	1
380–399	0
400–419	1
420–439	0
440–459	1
460–479	1
	234

tributions of the hypothetical universes from which the samples are conceived as drawn.

Let us consider the application to our material of the system of analysis invented by Karl Pearson. The invention was one of great beauty; anybody could be proud of having such an idea. Frequency curves are of three main types: The U-shaped distributions with modes at the two ends, the J-shaped ones with a mode at one end and the skew and symmetric distributions with a mode between the two ends at both of which the distribution sinks to 0. Can frequency curves be reduced to a single system? Pearson's answer is that by and large they can be. His original memoirs should be read, but as an initiation one may more easily read chapter XV of D. C. Jones's "First Course in Statistics." The central idea comes to this: That the various types of curve satisfy the differential equation

$$\frac{dy}{dx} = \frac{y(x+b)}{px^2+qx+r}$$

of which the integrals depend on five parameters (one being a constant of integration) and the types depend on various relations between the four parameters other than the constant of integration. Moreover just as the differential equation $dy/dx = -xy/\sigma^2$ of the normal or Gaussian curve may be obtained from the binomial expansion $(p+q)^n$ so the Pearsonian equation is related to the more general hypergeometrical expansion. It is altogether a magnificent conception. The actual fitting of the frequency function of the appropriate type is made from the values of the first five moments which are:

(1) Zeroth moment. $\Sigma y = N =$ total number of observations.
(2) First moment. $\Sigma yx/N = M =$ the mean value of x.
(3) Second moment. $\Sigma yx^2/N = M^2 + \sigma^2$ gives the standard deviation.
(4) Third moment. $\Sigma y(x-M)^3/N = \sqrt{\beta_1}\,\sigma^3$, related to skewness.
(5) Fourth moment. $\Sigma y(x-M)^4/N = \beta_2\,\sigma^4$, related to kurtosis.

Rules for making the fit and tables to aid in the calculations have been worked out by Pearson. The whole represents a very large amount of work. It may also be stated without much fear of contradiction that even with all the tabular aid provided, fitting some of these types and checking the fit is considerable of a chore.

Is it all worth while? In many cases I am afraid it is not. Howsoever happy I should have been to invent the original conception and to discover how well it worked out, I think that for applications such as the above I should have regretfully abandoned the system as unsatisfactory—as too much mathematics. The three series of infant mortalities have been fitted—I give you the fits taken from the literature. (I do not guarantee them; moreover the scale and the origin are not those of the table, *i.e.,* as they stand they are *not* the equations of the frequency functions—but that is of no moment for the present discussion.)

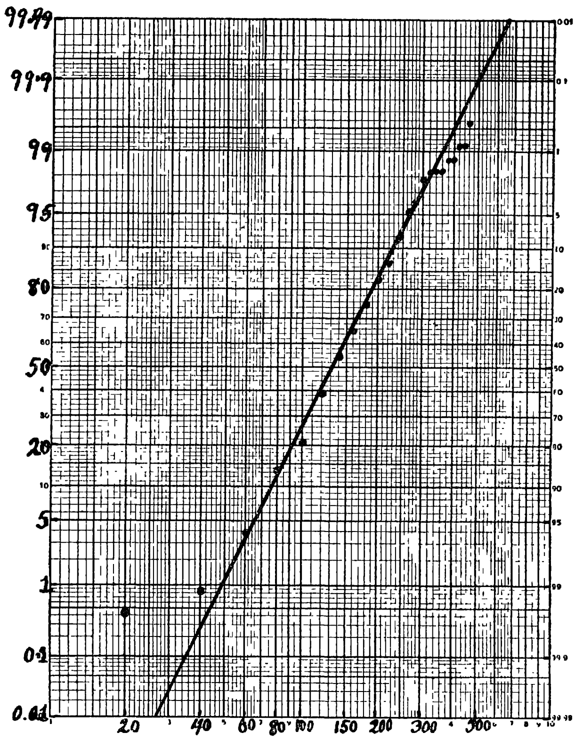
$$\text{I. } y = 46.5502\left(1+\frac{x}{2.8946}\right)^{4.4296}\left(1-\frac{x}{14.5155}\right)^{22.2126}$$

$$\text{II. } y = 0.005609\left(1+\frac{x^2}{(4.5095)^2}\right)^{-12.5407} \times e^{25.9862\tan^{-1}\frac{x}{4.5095}}$$

$$\text{III. } y = 4.2887 \times 10^{27}\,(x-20.9629)^{3.0627}\,x^{-19.9804}$$

Note that although the phenomena, infant mortalities, are of the same sort, the functional types of the frequency distributions are totally different. Moreover the functions are so complicated, taking into consideration not alone the type but also the numerical values, that any ready comparative interpretation of them seems out of the question. (It may be admitted that as mere empirical equations fitting the data they may perhaps be fair; at any rate the calculation involved in comparing them with the original data by Pearson's chi-square test for goodness of fit, is more than I should care to take unless there were some very important good to be had from the work; and finally the fit is obviously so bad at the ends that the X^2 – test would probably not be satisfied. I quote the results merely as an example of what I consider to be too much mathematics.)

You may ask how I should treat the data. It is a fair question. But I should have to retort with the question, what do you want to find out from the treatment? Much of our time may be saved from useless exercise of technique if we stop to ask what we are trying to find out and whether there is any expectation that we shall find it out by the application of any technique. But I will waive the question and set to work on the third set. First let us observe that infant mortalities under 20 are unlikely to be of statistical significance among the rural colored population; there are no such values in cities over 25,000 nor among rural whites, and I know of no reasonably large districts in the world where infant mortalities run under 20. Next at the other end of the distribution, mortalities above 400 may be open to suspicion, although there are places in the world, say in India or China, where such figures are not unknown, and maybe the same is true of some negro districts in America. However, we need not worry too much over a bad fit on the ends, and perhaps we shall be as well off anyhow as with equation III. As the distribution contains only positive values of the variable and is skewed toward the origin, the first trial may be made graphically on logarithmic probability paper which has been made familiar to us by Hazen and Whipple.[3] It is seen that the summation curve on this paper is practically straight (Fig. 1). There is evidence that



we have too many cases of mortalities under 40 (only 2, to be sure) and over 400; this is in accordance with our suspicions but should not be regarded as confirmation of them. Unless we are in an old field of science with well established fundamental laws, it is not through mathematical developments but by critical analysis of the sources of the data that confirmation of suspicions may be had. The type of frequency function therefore appears to be

$$dy = \frac{234}{\sqrt{2\pi}\,\sigma} e^{[\log(x/x_0)]^2 : 2\sigma^2} \quad d \log x,$$

where dy denotes the number of cases between two neighboring values of the logarithm of the infant mortality, x_0 is the geometric mean or the median of the values of x and σ is the dispersion or standard deviation of the values of log x.

The parameters of the curve may be determined graphically by methods that are well known and simple. For one such determination we have these figures.

The first 6 columns contain, respectively, the rate, the percentage less than that rate, the percentage in the immediate preceding interval of rate, the calculated frequency in that interval, the observed frequency and the differences between the observed and calculated values. The agreement is about all that could be desired. The last two columns give the calculation leading to X^2. The test is vitiated by the very low values of the theoretical frequency in the first single and last double interval; leaving these out we have 18 intervals with $X^2 = 14.3$. This represents a fit so good that chance fluctuations of sampling would produce a worse fit about twice out of three times. Such a correspondence between computed and observed values should be satisfactory and indicate that any more complicated curve such as III above is indeed entirely too much mathematics unless the fit is thereby much improved at the two extremes (as is not the case). The other tables of data can similarly be fitted adequately, I think, on logarithmic probability paper and spare us the annoyance of the complicated Pearsonian equations I and II.

FREQUENCY FUNCTION OF INFANT MORTALITY, 1918; RURAL COUNTIES, COLORED

Rate	% Less	% Diff.	Freq.	Obs.	Diff.	Diff.^{2}	X^2
20	.00	.00001	0.00234	1	+ 1.0	1.0	(427)
40	.24	.24	.6	1	+ 0.4	.2	.3
60	3.0	2.76	6.4	5	− 1.4	2.0	.3
80	11.3	8.3	19.4	22	+ 2.6	6.8	.3
100	24.8	13.5	31.6	19	−12.6	158.8	5.0
120	39.7	14.9	34.9	42	+ 7.1	50.4	1.5
140	54.0	14.3	33.5	37	+ 3.5	12.2	.4
160	66.3	12.3	28.8	30	+ 1.2	1.4	.0
180	75.5	9.2	21.5	19	− 2.5	6.2	.3
200	82.6	7.1	18.8	17	+ 0.2	.0	.0
220	87.5	4.9	11.5	10	− 1.5	2.2	.2
240	91.3	3.8	9.0	12	+ 3.0	9.0	1.0
260	93.9	2.6	6.1	8	+ 1.9	3.6	.6
280	95.7	1.8	4.3	4	− 0.3	.1	.2
300	97.0	1.3	3.0	2	− 1.0	1.0	.3
340	98.5	1.5	3.5	1	− 2.5	6.2	1.8
380	99.27	.77	1.8	1	− 0.8	.6	.4
420	99.66	.39	.9	1	+ 0.1	.0	.0
460	99.80	.14	.3	1	+ 0.7	.5	1.7
500	99.89	.09	.2	1	+ 0.8	.6	(3.0)
		99.89	234.1	234			$X^2 = 14.3$

The X^2 test of goodness of fit is due to Pearson and a fine contribution to statistics it was. More recently R. A. Fisher has been over the work and appears to have made some improvements in it and to have given valuable discussions of its application; he also has gone forward with the treatment of the error of the correlation coefficient. Perhaps I may say at this point that, so far as I can judge, R. A. Fisher is to-day the intellectual leader of the biometric school. He is not merely applying the method by rote but is thinking about the problems of statistical analysis in his own right. In his "Statistical Methods for Research Workers" there are several excellent things new with him; the book, however, gives merely

[3] See G. C. Whipple, "Vital Statistics," 2nd Ed., p. 508.

the result; for the detailed reasoning one must consult Fisher's original papers. To Karl Pearson, the great contributor of many important advances in probability and statistics and builder of the science of biometry upon the foundation of Galton, his master, it must be a source of genuine satisfaction, as he nears three score years and ten, to see near at hand so able a disciple as Fisher, an Elisha, who as so often in the history of science, is embroidering, and maybe patching up a bit, the mantle of Elijah ere it falls upon him.

It is interesting to note that the methods of fitting by moments and of estimating goodness of fit by X^2 are logically inconsistent.[4] This may be seen most easily on the zeroth moment which is the total number N of the observations. Let F be the observed frequencies and y the fitted values and let us consider for convenience the actual fit in the intervals tabulated —a discontinuous problem instead of the continuous problems—graduation only instead of graduation plus interpolation.

$$X^2 = \Sigma\frac{(F-y)^2}{y} = \Sigma\frac{F^2}{y} - 2\,\Sigma F + \Sigma y = \Sigma\frac{F^2}{y} - N$$

if we take $\Sigma y = N$. Suppose however that after making this fit we adjust the values of y to new values Z by applying a factor c so that $Z = cy$ and $\Sigma Z = cN$.

$$X^2 = \Sigma\frac{F^2}{Z} - 2\Sigma F + \Sigma Z = \Sigma\frac{F^2}{cy} - 2N + cN$$

The minimum value of X^2_Z will not be given by $c = 1$ but, differentiating, by

$$-\left(\Sigma\frac{F^2}{y}\right)\frac{1}{c^2} + N = 0 \quad \text{or by } c = \sqrt{\Sigma\frac{F^2}{Ny}} > 1$$

$$X^2 = N(c^2 - 1), \qquad X^2_Z = 2N(c-1)$$

The factor c and the change in X^2 are

$$c = \sqrt{1 + X^2/N}, \qquad X^2 - X^2_Z = N(c-1)^2$$

If X^2/N is well below 1, the results simplify to

$$c = 1 + X^2/2N, \qquad X^2 - X^2_Z = X^4/4N$$

and $\Sigma Z = cN$ becomes $N + X^2/2$. In the case above we should have a better fit by taking the total number of observations not as the actual number 234 but as $234 + 7 = 241$, but the improvement in X^2 would not be great. (However, the fit at the two extremes is so bad, small as are the actual numbers of cases, that X^2 if rigorously computed would obtain from the single observation in the 0–19 interval the value 427 which would make c about 1.7 and $X^2 - X^2_Z = 117$ or $X^2_Z = 327$, a great reduction, but still indicative of a very bad fit. Moreover, a change of N from 234 to about 400 is ridiculous, and although X^2 is thereby reduced the fit by any intuitive judgment is far worse.)

From such a simple consideration as this we obtain light on the significance of the X^2 test and yet more upon its limitations which are these: It is not a perfect abstract test but depends on the judgment of the person who uses it in respect both to the details of its arithmetic computation and to its interpretation as a criterion of goodness of fit; it is merely one factor, and a somewhat subjective factor, in aiding the investigator to make up his mind as to whether an attained fit is good enough or not. In the three cases of infant mortalities above mentioned my judgment based largely on graphical evidence and somewhat on a knowledge of the unreliability of the reported figures is that the simple solution I have given is as good as any reasonable person can ask, and this judgment is to my way of thinking not in the slightest called in question, but rather corroborated, by the application of the X^2 test. Many of the criteria of statistics are likewise of this loose character, they are not precisely mathematical truths as statistical criteria no matter how exact they may be as theorems in probability—and no other situation is useful or possible. For this reason I am not interested in a five-figure table of X^2, presumably two significant figures are all that are significant. It is so with many other matters and because it is I prefer that my students should not so much indulge by rote in elaborate but insignificant arithmetic as to form correct judgments by right of sound understanding. It is for this reason that I lament the low intellectual level of books and courses in statistics. It means too much of one kind of mathematics, too little of another kind.

EDWIN B. WILSON
HARVARD SCHOOL OF PUBLIC HEALTH

[4] Puristically speaking. A minimum problem determines its own criteria. Practically, if the fit is efficient (R. A. Fisher), the inconsistency is insignificant. If for simplicity I make so bold as to test the zeroth moment I should perhaps point out that ordinarily the number of observations N is not taken to be a disposable or fittable constant and that the theory of the X^2 test and the tabulated values of X^2 seem to depend somewhat on regarding N as given. Yet in some actuarial practices N is not preserved in the fitted graduation, it is not preserved by the ordinary least squares fit except when the type of fitted function y is of a restricted sort, and there seems to be no reason why a method of fitting appropriate to the theory of sampling might not be developed which should leave N disposable.

THE IMPORTANCE OF BIOLOGY FOR MANKIND[1]

PERMIT me to begin my short address by expressing my great joy to take part in such an important event

[1] Address delivered at the dedication of the new Charles Rebstock Biology Hall of Washington University, St. Louis, November 10, 1927.