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A MATHEMATICAL CRITIQUE OF SOME PHYSICAL THEORIES[1]

THE purpose of this paper was to review some of the mathematical-physical theories of the past and of the present, indicating briefly the nature of certain concepts upon which these theories rest as well as attendant logical difficulties, and proposing certain modifications. It goes without saying that geometry is the first and simplest of such theories. Some day, when the field of knowledge has extended so far that simplification becomes necessary, ordinary geometry may be approached somewhat as follows:

(1) Geometry treats of elements called *points* and the relation called *distance* between pairs of points.

(2) The complete tabulation of distances between pairs of points may be arranged as follows:

(a) the points P correspond to real number triples (x, y, z);

(b) the squared distance between P_1 and P_2 is $(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$.

All geometry follows very readily from these agreements. Beginning in this way one may successively define line-segments, lines, planes, perpendicularity, rectangular coordinate systems, etc. The whole body of geometrical fact with corresponding analytic framework is easily deducible, and yet one may stop at the fundamental principles without taking up beautiful but less vital geometrical studies. In its origin the geometrical concept of space is always to be associated with that of a corresponding body of reference.

Classical dynamics arises in the attempt to use Euclidean space and absolute time as the means for expressing the laws of nature. There lie certain fundamental difficulties at the very basis of this attempt to make space the container of matter. The simplest illustration of them arises in dealing with a collection of "equal rigid elastic spheres." When only two spheres collide, the assumed laws of contact action determine uniquely their directions and velocities after collision; but when more than two spheres collide, the situation is entirely different.

[1] Synopsis of address as retiring president of the American Mathematical Society before a joint meeting of the American Association for the Advancement of Science, the American Mathematical Society and the Mathematical Association of America. The full text will appear in an early number of the *Bulletin of the American Mathematical Society*.

Suppose that three equal spheres approach a point with equal velocities, the lines of motion being 120° apart and in the same place. If all three spheres collide at the same instant, considerations of symmetry alone demand that the spheres must rebound back along the same lines with a velocity equal to that of approach. But it is easily verified that if two of them collide ever so little before they collide with the third, the resultant motion will be decidedly different in character. Such a result seems to contradict the fundamental physical requirement of continuity. In fact the laws of action suffice to determine the behavior of two spheres which collide, but as more and more complicated simultaneous collisions between three or more spheres are considered, it is not possible to infer their behavior by an argument based on continuity or even symmetry, so that these laws need to be supplemented by indefinitely many others of arbitrary type, if the mathematical theory is to be determinate.

The situation is similar with n mass particles, attracting one another according to the Newtonian law; and the concept of elastic bodies so fundamental in classical dynamics presents even more formidable logical objections. For example, suppose that two equal elastic spheres under no pressure approach along their line of centers with equal velocities which exceed the disturbance velocity. The parts of the spheres which collide can not then rebound without interpenetration. Thus it appears as if the spheres are converted into a kind of lamina of infinite density moving radially outward in the plane of symmetry. But this yields a total change of state, which the theory of elasticity does not contemplate.

These illustrations show that the classical theory of particles, rigid and elastic bodies, needs to be supplemented by further conditions if the central difficulty of indeterminateness is to be avoided, and also that such further conditions will of necessity be artificial in character. The question now arises: Is it possible to conceive of simple laws of motion for systems of particles, and for continuous bodies in empty space, which will be unified and determinate? To secure such a system of particles it suffices to assume that in addition to the ordinary Newtonian force of attraction there is a repulsive force inversely proportional to the cube of the distance. Since the potential energy of the system then increases indefinitely when any two particles approach collision, it follows that collision can never take place. In order to deal with a continuous distribution of matter, we may assume that the law of force for the continuous distribution is the same as for the system of particles just considered. It is obvious that such a fluid can not contract indefinitely since then its potential energy would exceed the total initial energy; nor can it expand indefinitely unless sufficient kinetic energy is available. Evidently this fluid is entirely different in character from the elastic body under pressure, but it has at least the theoretical advantage of being free from indeterminateness. Two colliding bodies of this description will in general separate after a transitional period of interpenetration.

The chief mathematical instruments used by the physicists in dealing with space, time and matter in classical physics have been the Lagrangian and Hamiltonian equations. Poincaré proved it to be a general characteristic of equations of this type that small disturbances from stable periodic motion are essentially periodic.

It may be proved conversely that if a dynamical system is such that its state is determined by 2n coordinates, and if the perturbations from a periodic motion can be represented by trigonometric series, then the equations may be given Hamiltonian form.[2] Thus perhaps the only significance of the Hamiltonian form of equations in classical dynamics is to insure automatically that the perturbations of certain periodic motions are oscillatory.

The equations of Maxwell, giving the interplay between the electric and magnetic forces in space, have never been modified. The space-time background appropriate to this form is that of the special theory of relativity, in which space and time are taken relative to some reference body and in which the velocity of light appears as a characteristic limiting disturbance velocity.

In the simple case of a number of electrified particles, there will be an indefinite radiation outward of electromagnetic energy as oppositely electrified particles fall into one another, while those similarly electrified tend to separate indefinitely under the mutual forces of repulsion. Evidently such a system of electrified particles is of little physical interest.

The use of an elastic fluid under tension as the carrier of electricity seems at first sight to offer prospects of success. Further examination of the problem shows that it is impossible to secure stability of the kind desired, no matter what the relation between tension and density may be. But, aside from this instability, which arises from the fact that electricity of one sign exerts strong forces of repulsion upon itself, there is another difficulty which arises even in the consideration of neutral matter. In fact, there are two types of disturbance velocities, firstly, that of light and, secondly, that of the elastic wave of

[2] *Cf.* Paris *Comptes rendus*, September 20, 1926, and a paper which is about to appear in the *American Journal of Mathematics* for January, 1927.

the fluid. The same paradox might arise at collision as in the analogous classical situation.

The only possible elastic fluid would therefore seem to be one with a disturbance velocity equal to that of light at all densities. The fluid of this type may be termed the "perfect fluid," by analogy with the ordinary perfect gas. The expansive pressure is such a fluid is readily determined to be one half the density in absolute units, and so enormously great; on account of the relativistic character of this perfect fluid, the mass of a small part of it is not invariable but changes as the square of the density of the attached charge. Obviously the perfect fluid is a highly instable carrier of electricity.

The failure of attempts to make use of an elastic fluid as the carrier of electricity leads one to inquire whether it is not in the nature of the case that the elementary bodies such as the protons and electrons must have some sort of autonomous existence. Now the kinetic and elastic energy of the perfect fluid at low velocities can be defined, in such wise that the principle of conservation of energy holds. Let us suppose in addition that there is an individual "atomic potential" energy of positive volume density, ψ, where ψ has a value fixed for all time at each point of the fluid. This leads to a supplementary body force, proportional to the gradient of ψ in space, and also to a surface pressure inward, proportional to ψ. In this way indefinite expansion is prevented, for it would involve an indefinite increase in the atomic potential energy.

At first sight this seems to insure a stable spherical form of equilibrium. However, further examination shows that the nucleus is amorphous under radial displacement. But now suppose that the protons are made of very small parts of the fluid with charge $+e$, while the electrons are also made of parts of the fluid carrying the charge $-e$, both with suitable atomic potentials. Let us suppose furthermore that such an electron can be penetrated freely by the proton. Under these circumstances there will be a stable spherical form of equilibrium, in which the proton coincides with the electron; the tendency towards amorphous shape of the electrons and protons will be destroyed by the attractive forces between them.

Here perhaps is a kind of two substance theory of matter and electricity which will be found to meet the fundamental mathematical requirements of determinateness and stability.

The space-time framework of general relativity is adapted to the concept of atomic potential; for this purpose the energy tensor T_{ij} may be defined as consisting of the elastic and electromagnetic energy tensors due to the protons and electrons, and of a further term, ψg_{ij}, where ψ is the atomic potential.

If we grant the four-dimensional nature of space-time, the argument of continuity seems to make it imperative that the atom is an oscillating electromagnetic system. The central facts about the atomic oscillator are essentially two: first, it acts like a number of simple resonators of perfectly definite frequencies, such as those given by the Balmer formula in the case of the hydrogen atom; and secondly, these frequencies are excited only by means of certain quanta of energy. Now there need be no essential difficulty in accounting for this second fact. Imagine a pendulum to swing in a viscous medium whose viscosity diminishes rapidly as the distance from the position of equilibrium increases. Only with sufficient initial velocity will it oscillate back and forth, traversing the viscous region in damped hormonic motion. Consequently it is possible to conceive of the so-called "energy levels" as defining the amount of energy necessary to carry the oscillators so far from equilibrium that they will move back and forth past the position of equilibrium. Thus the first and most fundamental task appears to be to find an oscillator possessing the desired frequencies. Afterwards one may investigate in detail the rate of electromagnetic radiation, which may correspond to viscosity.

In particular it will be of interest to consider the small oscillations of the fluid proton and electron as specified.[3] The three equations determining the frequencies are analogous in type to the "wave equation" of Schrödinger.

It seems to be of decided importance to develop theories, like the above, which meet the elementary mathematical demands of actual determinateness and stability. This does not seem to have been done in a single case hitherto.

In conclusion the statistical properties of non-Hamiltonian equations were referred to. These seem to be best suited to represent atomic systems taken as possessing a finite number of degrees of freedom; for such equations only can there be a set of periodic motions to which every other motion is in general very near.[4] Hence such differential equations may yield the effect of quantum orbits without any quantum conditions.

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[3] A development of the theory outlined will be published in a forthcoming number of the *Proceedings of the National Academy of Sciences*. On proper choice of the "substance coefficients" and "atomic potentials," the theory leads to a formula of Balmer type for the frequencies.

[4] *Cf. Göttinger Nachrichten,* 1926.