THE RELATION OF THE RESTRICTED TO THE GENERAL THEORY OF RELATIVITY AND THE SIGNIFICANCE OF THE MICHELSON-MORLEY EXPERIMENT

THE SIGNIFICANCE OF THE VELOCITY OF LIGHT IN THE GENERAL THEORY

IT is customary to regard the restricted theory of relativity as contained in the general theory as a special case applicable to regions far removed from matter. On this line of thought the quantity c, the velocity of light, makes its appearance very early in the discussion of the general theory, and with the status which it derives from the restricted theory. This results in an apparent dependence of the general upon the restricted theory which is much stronger than the exigencies of the situation would require. For this reason the following line of approach may have some advantages.

The analysis of course follows the well-known lines, all that is here attempted being a modification of the places where emphasis is placed on certain matters. We start with the Einstein equations

$$G_{\mu\nu} = 0 \tag{1}$$

representing the vanishing of the contracted Riemann-Christoffel tensor. The problem is to find a solution of these equations for the $g_{\mu\nu}$ which will cause the law

$$\delta \int ds = 0 \tag{2}$$

or, what is the same thing, the set of equations

$$\frac{d^2x^\alpha}{ds^2} + \left\{\mu\nu \cdot \alpha\right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{3}$$

to lead to results consistent with the facts of planetary motion.

The Schwarzschild solution of (1) for symmetry about a point[1] gives $g_{\mu\nu}$'s which make the line element of the form

$$ds^2 = -\gamma^{-1}dr^2 - r^2d\vartheta^2 - r^2\sin^2\vartheta d\varphi^2 + c^2\gamma dt^2 \tag{4}$$

where

$$\gamma = 1 - \frac{2m}{r} \tag{5}$$

m is a constant; and, as far as the requirement that the g's shall be solutions of (1) is concerned, c can be any constant whatever.[2] In the customary demonstrations the units in which dt is measured are supposed chosen in such a way that c^2 is unity. In order to emphasize the salient features appropriately, however, we have retained the c; for, if the method of measuring dt is provisionally assigned, the value of c becomes automatically determined by the condition imposed by planetary motion.

Following the usual procedure[3] of building up the equations of motion from (3), we arrive at

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2 \tag{6}$$

$$r^2 \frac{d\varphi}{ds} = h \tag{7}$$

where $u = 1/r$ and h is a constant for the orbit. Integrating (6), with approximations for convenience (though not of necessity) we arrive again, in the usual way, at

$$u = \frac{m}{h^2}(1 + e\cos(\varphi - \omega - \delta\omega)) \tag{8}$$

where e and ω are constants for the orbit, but

$$\delta\omega = \frac{3m^2}{h^2}\varphi \tag{9}$$

Apart from the small term $\delta\omega$, an examination of the orbit would serve to determine only m/h^2, and not m and h separately. The term $\delta\omega$, however, serves to fix (m^2/h^2), so that by its inclusion both m and h are determined from experimental observations. The determination of h serves to determine dt/ds for any particular method of measuring dt. For, (7) is equivalent to

$$r^2 \frac{d\varphi}{dt}\frac{dt}{ds} = h \tag{10}$$

and since ds is given by (4), and m and h are known, the value of c necessary to correspond to the facts can be evaluated.[4] Without knowledge of h it would be impossible to utilize (10) to determine c. The quantity c thus finds its natural origin in the departure of the motion from the Newtonian law, and it plays no part in the expression of the principal part (the Newtonian part) of the motion.

But there is something else. The quantity c represents the maximum velocity (in the sense defined by

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ )$$

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[1] See, for example, Eddington's "The Mathematical Theory of Relativity," pp. 82–85.

[2] For the benefit of those who are not specialists in the subject and who, on referring to Eddington's book, for example, find the c^2 here referred to missing, it will be sufficient to call attention to the fact that its inclusion will leave the $g_{\mu\nu}$'s, as given by (4) with (5) solutions of (1), as may readily be seen by checking through the analysis, or, more obviously, by observing that the inclusion of c^2 merely amounts to a change of the units in which dt is measured.

[3] See Eddington, *loc. cit.*

[4] Formally, both m and h are necessary for the purpose since ds involves m through γ. As a matter of fact both m and h are available, but practically speaking it would be sufficient to determine c from (10) by writing $ds^2 = c^2dt^2$.

which any planetary particle can attain. This will probably be sufficiently observed from the fact that for all values of ds^2 greater than zero, (4) gives

$$dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 < c^2 dt^2$$

since $\gamma < 1$. The fact may be proved more formally and generally, but in view of customary recognition of its truth we shall not elaborate further on the proof.

The existence of a maximum value c for the velocity attainable by a particle reminds us of the statement that no particles can move faster than the velocity of light, so we are tempted to inquire whether the velocity of light (understood as measured far from matter, of course) will serve for the quantity c, which had its origin in the deviations from the Newtonian law of planetary motion. We find that the velocity of light will so serve.

Again, since the maximum velocity which a particle can attain is equal to the velocity of light, we are tempted to try whether a ray of light will follow the path of a particle which (at infinity of course) travels with the velocity of light. In this way we are led to the idea of the bending of light by the sun in a manner which may appear more suggestive to some than one which leads to it through the Principle of Equivalence.

The Relation of the Restricted to the General Theory

At an infinite distance from matter, the line element (4) assumes the form

$$ds^2 = -dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + c^2 dt^2$$

Or, in rectangular coordinates

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2 \qquad (11)$$

It is now a matter of algebra that the quantity ds given by (11) is invariant under the Lorentzian transformation

$$x' = \beta(x - vt) \quad y' = y \quad z' = z \quad t' = \beta\left(1 - \frac{v}{c^2} x\right) \qquad \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This does not necessarily carry with it the restricted theory in the sense that observers moving relatively to each other with constant rectilinear velocity will automatically adopt systems of measures related by the Lorentzian transformation. Nor does it imply that the laws of nature are necessarily invariant under that transformation, except in the spirit of the general theory itself, which calls for an expression of the laws in a form invariant under *any* transformation—a requirement which concerns itself with a different mode of expression of the laws, however—one of wider generality and therefore one aiming toward a less specific expression than that contemplated in the restricted theory. In fact, the general theory can stand without the restricted theory used in the foregoing senses.

The symmetry of the line element at infinity (as elsewhere) follows, of course, from the fact that a symmetrical solution of (1) was sought intentionally; and, it would follow that any one, who as a result of infinitely accurate measurements found the line element (4) satisfactory in its expression of the g's as functions of his coordinates near the center of symmetry and on out therefrom, would automatically be driven to the conclusion that far from the center of symmetry that line element assumed (again in terms of his coordinates) the form (11). By carrying over to this limiting case the law $\delta \int ds = 0$ for the path of a ray of light, he would be led to conclude the constancy of the velocity of light in all directions at infinity. However, a transformation[5] of coordinates from the r, ϑ, φ, of equation (4) to coordinates r', ϑ', φ', differing from there by terms of the order v^2/c^2 (where v is a quantity of the order of magnitude of what, in pre-relativity days, was regarded as the earth's absolute velocity) would produce a departure from symmetry in the line element, which would only react on the equations of planetary motion deduced from (3) to an extent which was almost immeasurably small.[6] A line element of this type would revert at infinity to a form which differed from the symmetrical form by terms of the order v^2/c^2. It is of course true that the set of coordinates in terms of which the unsymmetrical line element is expressed, which set we shall call set B, is transformable (but not by the Lorentzian transformation) into the set A in terms of which the line element at infinity assumes the form (11); but it might be that our actual measures correspond to the set B and not to the set A. In this case, the law $\delta \int ds = 0$ for a ray of light would not lead to equal velocities in all directions, even at infinity.

[5] We do not wish to assume a *Lorentzian* transformation.

[6] Before the days of general relativity, expressions involving v^2/c^2 (where v is here the relative velocity of the sun and a planet) were deduced on the basis of the restricted theory of relativity. They gave results differing from the Newtonian results by amounts in general beyond the limits of observational error (see paper by de Sitter, Monthly Notices of Roy. Astr. Soc., Mar., 1911, p. 388). The largest effect, that concerned with the perihelion motion of Mercury, amounted to only 7 seconds of arc per century, and this was for a magnitude of v corresponding to the relative velocity of Mercury and the sun, a magnitude considerably greater than the values of v under discussion in the Michelson and Morley experiment. The effect of v on the bending of light by the sun would be, of course, entirely negligible.

Now, even as regards the restricted theory itself, the point of most important significance has but little to do with how the measures of different observers are related to each other. Its import lies rather in the assertion that the laws of nature far removed from gravitational fields are such that there is *some* system of coordinates in which they can be expressed in a form which will remain invariant for any transformation of coordinates of the Lorentzian type. The test here involved is a matter of algebra and has nothing to do with the question of whether an observer would automatically assume one set of the group of coordinate systems in terms of which the laws were invariant for the Lorentzian transformation. Now it might happen that when expressed in terms of the coordinates A the laws were invariant under the Lorentzian transformation. In this case, if they were correctly expressed in terms of the set B, they would not be invariant under this transformation. In the set of coordinates B we should not find the velocity of light the same in all directions. Our theory of the phenomena, expressed in these coordinates, would not predict it. In the set A we should find the velocity the same in all directions, and, in this set, or in all sets derivable from it by a Lorentzian transformation, we should have the features of the restricted theory which are of most value, the features which make this very criterion of invariance a test of the validity of a law. This frame of reference, in which the set of coordinates used by the observer was the set A, would serve as an absolute system of coordinates.[7]

Now, if the set of coordinates B happened to differ but little from the set A, it would follow that the laws of motion expressed in terms of B would be very approximately, though not exactly, invariant under a Lorentzian transformation (a mathematical transformation, not a physical change of measures). They would only be in error to the extent corresponding to the substitution of the coordinates of the B set in place of those of the A set, and the difference between the A and the B set would only amount to the almost immeasurably small difference which would correspond to magnitude of the effect which was anticipated in the Michelson and Morley experiment in pre-relativity days. While the effect of this difference would be enhanced when the substitution of the coordinates was made, for example, in the law of motion of a particle for a case when it was traveling with velocity nearly equal to that of light, the effect would always be far less than that resulting from the unavoidable errors in the measurements.

To summarize the situation, it may be said that on the above views the structure of such laws of physics as the laws of electrodynamics, carrying with them as they do invariance under the Lorentzian transformation for their own case, would continue to have full significance in implying such invariance for all the laws of physics when expressed in terms of a suitable coordinate system. The only new element in the line of thought lies in the possible belief that the actual coordinate systems which we use differ slightly from those in terms of which the laws should be expressed for their exact truth, the difference being so small as to have failed to cause us to detect that these laws were not quite true in terms of our coordinates except when, as in the case of the Michelson and Morley experiment, the test can be made with such precision as to make the difference between the two sets of coordinates play a vital part.

On the above line of thought the significance of the restricted theory would thus lie in the statement that there exist systems of coordinates in terms of which, far removed from matter, the line element assumes the form (11), and the *forms of the laws* of nature, when expressed in terms of these coordinates, are invariant for that type of transformation (the Lorentzian transformation) which leaves the *form of the line element invariant.*

Much of the foregoing discussion is of course prompted by the recent experiments of Professor D. C. Miller, in which a positive effect is claimed for the Michelson and Morley experiment. The object here is not to attempt any discussion of these experiments themselves. Much is undoubtedly to be said from the standpoint of possible effects which might be classed under normal disturbing influences; but it is hardly appropriate to discuss this feature before the experimental results have been published in full. The object here is simply to consider the status of the restricted theory of relativity if it should turn out that a positive effect remains in this experiment and in the various other experiments which have been performed to test the theory of relativity, after all disturbing effects (in the ordinary accepted sense of the term) have been removed. The attempt has been made to make the discussion in terms of the concept of a difference between the actual coordinate system used, and an ideal system in terms of which, with its related transformed systems, the laws of nature might be correctly expressed in a form invariant under the Lorentzian transformation. The fact that Professor Miller finds results which depend upon the altitude of the apparatus[8] would necessitate the assumption

[7] It might happen that by changing the motion of the system of measures B we could make them revert automatically to the system A. In this case we could define a meaning to an absolute velocity of the system B.

[8] Results which it is difficult to harmonize with Professor Michelson's experiment on the independence of ether drift on altitudes, *Amer. Jour. Sci.*, No. 3, p. 475, 1897.

that the measuring system also depends upon proximity to the earth's surface.

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A YEAR OF PROGRESS FOR ORGANIZED MUSEUMS

Two years ago The American Association of Museums secured pledges amounting to nearly $30,000 annually for three years, and established at Washington national headquarters which subsequently were removed to New York City. The work has advanced rapidly as indicated by the report for the year just closed—the second year of operations—which shows income of $120,000.

The short period which has elapsed since the enlarged program was undertaken has witnessed development of the various services which were projected at the outset, but experience has served to shift a part of the emphasis from service to independent researches and promotions. A number of such projects have been planned and financed successfully, and some have already been brought to completion. The most significant elements of the year's progress are felt to be such pieces of work that can stand alone.

The Yosemite Museum

An outstanding accomplishment has been the building of a museum in Yosemite National Park. In June, 1924, the association's Committee on Museums in National Parks made application to the Laura Spelman Rockefeller Memorial for a grant to make possible the erection of a museum building in Yosemite Valley, and also to provide for installation of exhibits and staffing of the museum for three years, during which period its maintenance might be absorbed by the government. At the same time a small sum was requested for an investigation of museum needs in other national parks and the development of a comprehensive program.

On July 11, the memorial voted $70,500 for building, equipment and maintenance, and appropriated an additional $5,000 to the committee for its own work. Dr. Hermon C. Bumpus as chairman of a sub-committee made a trip to California and under his general supervision the work has been carried forward at so rapid a pace that within ten months the building has been completed and the preparation of exhibits far advanced.

Reports of European Surveys

Director Charles R. Richards has completed a report of a survey made last year of industrial museums in Europe. The manuscript is in the hands of the publisher and will appear as a book entitled "The Industrial Museum." During the year, situations have developed in New York City and Chicago that promise early opportunities to apply the results of this study.

A report on industrial art museums in Europe, another investigation which was made last year by Professor Richards, is well advanced.

Promotion of Small Museums

During the year Mr. Coleman has made three field trips which have enabled him to visit more than 200 museums in 85 cities in 24 states, from coast to coast and from Canada to Mexico. This survey was made possible by a grant from the Carnegie Corporation of New York and its purpose was to determine the conditions of museums in small communities.

This field work made plain the need for a comprehensive handbook of museum methods. Accordingly the secretary undertook the preparation of such a book, and the manuscript is now practically complete. The "Manual for Small Museums" will appear shortly as a book of some 45 chapters.

The development of this work has served to crystalize a program for the promotion of small museums, and has also offered many opportunities for local service.

Financing of New Projects

Five new undertakings have received grants during the year. The General Education Board appropriated $21,000 for the expenses of an official commission to the International Exposition of Industrial and Decorative Arts which is now being held in Paris, $10,000 to bring back from the exposition and to exhibit in the principal museums of this country a representative collection of the finest examples of European decorative and industrial art and $1,000 to develop and circulate collections of the best examples of American textiles, ceramics and other objects of industrial art. The Carnegie Corporation of New York has appropriated $1,500 for publication of the "Manual for Small Museums" and $2,500 for a study of museum fatigue.

The above-mentioned commission to Paris was appointed by Secretary of Commerce Hoover, with Director Richards as chairman. The study of museum fatigue has been placed in the hands of Professor Edward S. Robinson, of the University of Chicago, who has developed the outlines of an investigation to be carried forward with the help of the Art Institute of Chicago.

Committee Work

Besides the Committee on Museums in National