the universities are beginning to be led by the industries, instead of vice versa. But that will not last long. The universities will lead again, I am sure of it.

The industries are anxious that the universities should do their work of training research men as well as it possibly can be done. I heard one captain of industry remark one day that, perhaps, it would be a good thing if the industries would set aside a part of their profits, derived from the development of scientific research, for the benefit of universities, to enable them to give better and better instruction in scientific research. I also know that some of the industries are subsidizing some of the university laboratories for carrying on certain research work; not developmental work, but purely scientific research work. The best of the industries are not trying to debase the real research work of universities by giving them problems which are nothing but technical development work. The industries can do that themselves. What they would like to see the universities do is to carry on pure scientific research work, and to produce young men who have a truly scientific mental attitude. This cooperation between the scientific work in the universities and in the industries has already produced wonderful results, and it will produce more and more, and I am quite sure that some day the achievements from this cooperation will prove even to the most ordinary type of mentality that the best work can be done only by experts who have the proper training. That is the doctrine which we need in this country, and if it is adopted, not only in the industries, but in every activity of government, then the prophecy of Rowland will be fulfilled. I am sure that it will be adopted some day, because that is one of the best ways to make democracy safe for the world.

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THE FOUNDATION OF THE THEORY OF ALGEBRAIC NUMBERS[1]

II

We shall next see that certain modifications are to be introduced in order that the usual theorems of arithmetic hold true in the more general realms. For example, in the very simple realm that exists by adjoining $\sqrt{m}$ to the usual realm, it may be proved when m is greater than 3 and is not a perfect square that the Euclid Algorithm is not applicable, and there is no such thing as the greatest common divisor in the usual sense. By way of illustration observe that in the realm $R(\sqrt{-5})$ we have $21 = 3.7 = (4+\sqrt{-5})(4-\sqrt{-5}) = (1+2\sqrt{-5}) \times (1-2\sqrt{-5})$, where all the factors are irreducible integers. Thus it is evident that the factorization of an integer into its irreducible (or prime) factors is in these extended realms *not* a unique process as is the case in the usual realm of arithmetic and as is also true in the realms $R(i)$ and $R(\omega)$. And here is the difficulty that mathematicians at first found perplexing, a difficulty which it was necessary to overcome before the laws of arithmetic could be regarded as universal.

The problem may be recast as follows: Let ρ be the root of an algebraic equation of the nth degree whose coefficients belong to the usual (natural) realm of rationality and let ρ be adjoined to the usual realm. We thereby create an algebraic realm $R(\rho)$ of the nth degree. *Determine the arithmetic of this extended algebraic realm.*

By making use of the above example we shall anticipate the results that follow, particularly those that are connected with the *Theory of Ideals*. The reader is thus enabled to see the trend of the later theory and with this in view he is asked to accept without proof the statements given immediately below.

Write $T_1 = (3, 1+2\vartheta)$, $T_2 = (3, 1-2\vartheta)$, $T_3 = 1, 1+2\vartheta)$, $T_4 = (7, 1+2\vartheta)$ where $\vartheta = \sqrt{-5}$. It may be proved by taking the products of the ideals that

$$T_1 T_2 = (3),\ T_1 T_3 = (1+2\vartheta),\ T_1 T_4 = (4-\vartheta),$$
$$T_2 T_3 = (4+\vartheta),\ T_2 T_4 = (1-2\vartheta),\ T_3 T_4 = (7).$$

None of these quantities is a unit in $R(\vartheta)$ and they are all prime ideals since, if N denotes the *norm* of an algebraic quantity and that is the product of the quantity and its conjugates, so that $N(T_1) = (3, 1+2\vartheta)(3, 1-2\vartheta)$, then is

$$N(T_1) = N(T_2) = 3;\ N(T_3) = N(T_4) = 7.$$

Thus it is seen that the factorization of 21 into its prime ideal factors, namely, $21 = T_1 T_2 T_3 T_4$ is a unique process. It is also seen that the different methods of factorization given above for the integer 21 in the realm $R(\vartheta)$ are had through the different combinations in pairs of the T's.

It thus appears that the prime ideals in this extended realm take the place of prime integers in the usual arithmetic; and one of the objects before us is to establish what is the historical origin of these prime ideals, as well as to study what they are.

Returning to the discussion of the proof that the Greater Fermat Theorem does not admit integral solutions, consider the simple case

[1] Concluding part of the address of the vice-president and chairman of Section A—Mathematics, American Association for the Advancement of Science, Washington, December 31, 1924.

$$(1) \qquad z^4 = x^4 - y^4 = (x-y)\ (x+y)\ (x-iy)\ (x+iy) = T_1\ T_2\ T_3\ T_4,$$

and observe that the factors on the right-hand side are irreducible in R(i). Since the complex quantities in this realm obey the same laws as do the real quantities in their realm, we may derive a *correct* proof that equation (1) *can not* be solved in real integers.

Consider next the equation

$$z^n = x^n - y^n = (x-y)\ (x-\alpha y)\ (x-\alpha^2 y) \ldots (x-\alpha^{n-1}y) = T_1 \cdot T_2 \ldots T_n,$$

where α is a primitive root of $x^n = 1$. The T's being quantities of the realm $R(\alpha)$, are of the form

$$T = a_0 + a_1\alpha + a_2\alpha^2 + \ldots a_{n-1}\alpha^{n-1},$$

where the a's are rational integers.

There are now two questions before us. 1°: Are the factors T irreducible in $R(\alpha)$? It is found that they are. 2°: Is this factorization unique? It is found that it is *not*. According to the testimony of Guldenfinger and of Grassman, Kummer (1810–1893) proved the Fermat Theorem as to the requirements of the first condition and submitted his MS about (1843) to Dirichlet. Dirichlet pointed out that the second condition must also be satisfied.

In 1844 in his celebrated paper, "De numeris complexis, etc.," Kummer wrote,

> Maxime dolendum videtur, quod haec numerorum realium virtus, ut in factores primos dissolvi possint, qui pro eodem numero semper iidem sint, non cadem est numerorum complexorum, quae si esset, tota haec doctrina, quae magnis adhuc difficultatibus laborat, facile absolvi et ad finem perduci posset.

However, in a letter to Liouville (April, 1847) (see *Journal de Mathématiques,* Vol. 12, p. 136) Kummer again wrote,

> Quant à la proposition élémentaire, qu'un nombre composé ne peut être decomposé en facteurs premiers que d'une seule manière, je puis vous assurer qu'elle n'a pas lieu généralement tant qu'il s'agit de nombres complexes de la forme T [defined above] mais qu'on peut sauver en introduisant un nouveau genre de nombres que j'ai appelé nombre complexe idéal.

Thus it was Kummer who, in this maze of doubt and uncertainty, found a means of overcoming the difficulties and dilemmas that had been encountered. By the introduction of the prime ideal factors it was seen that the rational prime integers are no longer the extreme elements in the extended realms of rationality. Although Kummer's principles had to do for the most part with the algebraic numbers which are derived from the roots of unity, the ideal numbers which he introduced served as a guide for the general theories that were soon afterwards invented.

Among others who also were working in the theory of algebraic numbers that are formed from the roots of unity may be mentioned Jacobi, Cauchy and Eisenstein.

L. E. Dickson in his "History of the Theory of the Theory of Numbers," Vol. II, p. xix, writes:

> Although Gauss had proved in 1832 that the laws of elementary arithmetic hold also for complex numbers (numbers like $5 + i7$) and made a brilliant application of them in his investigation of biquadratic residues, the theory of algebraic numbers was really born in 1847. For it was then that the mathematical world became definitely conscious of the fact that complex integers (as T above) do not obey in general the laws of elementary arithmetic. This historical fact came to light through discussions of lacunae in the attempted proof of Lamé that if n is an odd prime, $x^n + y^n = z^n$ is not satisfied by such complex integers. Other errors of the same nature were made by Wentzel and by so great a mathematician as Cauchy. Curiously, Kummer himself made the error, in a letter of about 1843 to Dirichlet, of assuming that factorization is unique, so that his initial proof of Fermat's Theorem was incomplete. But Kummer did not stop with the mere recognition of the fact that algebraic numbers do not obey the laws of arithmetic; he succeeded in restoring the laws by the introduction of ideal elements, this restoration of law in the midst of chaos being one of the chief scientific triumphs of the past century.

We are pleased to add that Dickson himself has made some far-reaching discoveries in this same field which must give him in the mathematical world the recognition accorded to Kummer, Dirichlet, Dedekind and Minkowski.

The theory of analytic functions was developed in the French school. Dirichlet became well versed in this subject during his stay in Paris and upon his return to Germany through his lectures, particularly on the partial differential equations, established in the German school a theory that was already well known in France, due to the efforts especially of Poisson, Fourier, Ampère and Monge. Applying his knowledge of analytic methods to the problems that arise in the consideration of complex units, Dirichlet was able on the one hand to establish a *fundamental* system of units for any algebraic realm and on the other hand he was able to derive a formula for the presentation of the number of classes into which the algebraic numbers of a realm may be distributed. These two principles must be included in any arithmetic of algebraic numbers as we have already indicated. And thus, as Kummer has said, Dirichlet made an epoch in this theory as Descartes had done in the application of analysis to geometry.

Riemann learned the analytic method from Dirichlet and this led him far into his geometrical researches regarding which we have already spoken. Thus it is seen that the work of the French school is basal in many of the results that have been indicated in this paper.

We are now brought face to face with two other disciples of Dirichlet, namely, Kronecker and Dedekind. Both of these men felt the necessity of generalizing the notion of the ideal factors for the case of any algebraic quantities, and that is, of any quantity of an algebraic realm however generalized. Observe that the ideal numbers of Kummer had to do with cyclotomic (circular) realms, which are had through the adjunction of the roots of a binomial algebraic equation to the usual realm of rational numbers. These ideal numbers in the form presented by Kummer are susceptible of simplification and generalization.

Frobenius in his "Gedächtnissrede auf Leopold Kronecker" writes:

> The genius of Gauss in the treatment of the cyclotomic numbers (roots of unity) make algebra pay tribute to arithmetic and Jacobi's conquering strength lays at her (arithmetic's) feet the measureless treasures of formulas from the theory of elliptic functions and into her service forced the finest methods of analysis. It is Kronecker's everlasting service that he made this self-sufficient science become a servant to both algebra and the Theory of Functions.

And Kronecker in his "Antrittsrede" to the Berlin Academy said: "Die Verknüpfung dieser drei Zweige der Mathematik erhöht den Reiz und die Fruchtbarkeit der Untersuchung."

The investigations of the complex numbers formed from the roots of the Abelian equations led Kronecker to the algebraic-arithmetic problem of forming all Abelian equations for any realm of rationality. The solution of this problem he communicated to the Berlin Academy in 1853.

From this time on Kronecker laid especial emphasis upon the treatment of algebraic questions from an arithmetical point of view and in the investigation of such problems he entertained the idea of extending Gauss's conception of congruences with respect to a rational integer as modulus to the conception of congruences with respect to an arbitrary system of moduli, a conception which in its incipiency had already been conceived by Serret and by Schönemann. In the preface to Vol. I of Kronecker's Works, Hensel writes:

> Under the name of *General Arithmetic* Kronecker understood the application of the conceptions and methods of the Theory of Numbers to the investigation of rational functions of any number of variables. This greatly extended field of investigation embraces the consideration of systems of integral numbers, the entire field of the theory of numbers, the investigation of linear systems, the theory of determinants, bilinear and quadratic forms and finally the general field of algebraic numbers and functions of one and of several variables.

Kronecker employed the systematic application of indeterminate coefficients in the definition of the ideal quantities and by using several variables in the formation of his functions he attempted to overcome many of the difficulties and to avoid many of the imperfections that are experienced in the use of one variable. Instead of the association of higher kinds of algebraic irrationalities he widened the dimension of the original realm of quantities by the introduction of forms of several indeterminates and thus he gave essentially new points of view for the realms of rationality which through such adjunctions contain not only numbers and functions of one variable but also functions of several variables; and through a finite number of integral algebraic quantities he was able to express all such quantities of the realm. Kronecker wished to see in the greatest common divisor of several integral quantities *not* the only thing common to such quantities. This he held is a common divisor of the first kind (Stufe). There are with him common divisors of a higher kind. The general method for the treatment of such quantities Kronecker presented in a condensed and exceedingly difficult form in a memoir entitled "Grundzüge einer arithmetischen Theorie der algebraischen Grösse," which he dedicated to his friend and teacher Kummer on the commemoration of the latter's seventieth birthday. The work may best be described in Kronecker's own words regarding Legendre's "Théorie des Nombres:" "It can not be regarded as a well-ordered and well-arranged work."

From what has been seen Kronecker wished to treat in its generality every branch of mathematics under the one heading "General Arithmetic," a theory which divested of its analytic properties should rest upon something akin to the rational integers as its final substructure. A systematic treatment of Kronecker's ideas with the natural extensions, ramifications and applications and not in the form of a *general arithmetic* or a *general analysis* is a work well worth doing by some capable young American.

As a rule great discoveries do not fall out of clear skies. It is usually with much patient wooing attended by prolonged labor and arduous toil that they are produced. Newton had his precursors and the theory of fluxions was not the sudden output of a fertile brain. Fortuitously, the mathematicians mentioned above with possibly others served in guiding aright the one whose mathematical-philosophic genius gave an easy and comprehensive method for the treatment of algebraic numbers in all their generality. Richard Dedekind (1831–1916) devised, systematized and extended this theory from time to time and modestly incorporated his results as the "Eleventh Supplement of Dirichlet's Zahlentheorie." This excellent work was edited through several editions by Dedekind.

I produce here briefly the outlines of Dedekind's discoveries which, resting upon the deeply imbedded bases already considered, are to be regarded as the firm foundations of the theory of algebraic numbers.

Let a and b be any two fractional or integral numbers in the usual realm of rational numbers. Observe that the linear form $ax + by$, for integral values x and y, represent all those rational numbers that are divisible by the greatest common divisor of a and b. We may therefore say that any number t is divisible by the complex of numbers a and b (which complex denote by [a, b]), if it is possible to determine two rational integers x and y such that $t = ax + by$. This is an extension of the ordinary conception of divisibility in that t is divisible by a if $t = ax$, where x is an integer. This extension is clearly superfluous, so long as we remain in the usual realm of rational numbers; for in this case every number that is divisible by the complex [a, b] is divisible by the greatest common divisor d of a and b, and reciprocally, every number that is divisible by d is divisible by [a, b]. Accordingly, we may write $d = [a, b]$.

It is quite otherwise if we extend the realm of rational numbers to an algebraic realm Ω. The following definition is accordingly introduced: *The integral or fractional number λ is said to be divisible by the complex $[\alpha, \beta]$, if there exist two integers ξ and η such that $\lambda = \alpha\xi + \beta\eta$ where all quantities belong to Ω.*

This conception is no longer superfluous. For, if δ is a quantity through which both α and β are divisible, then every number that is divisible by $[\alpha, \beta]$ is clearly divisible by δ. However, every number that is divisible by δ is *not* divisible by $[\alpha, \beta]$. For, if this were true, then δ would itself be divisible by $[\alpha, \beta]$ and would accordingly be expressible in the form $\delta = \alpha\xi_1 + \beta\eta_1$, where ξ_1 and η_1 are integers in Ω. Hence δ would be the greatest common divisor of α and β in the sense that is usual in the theory of rational numbers. If, however, Ω is an arbitrary algebraic realm of rationality, we meet with a difficulty. For, if in the definition of *divisibility* we limit the discussion to a definite realm and if we define divisibility as we have just done for the rational numbers, there is no greatest common divisor in general for two numbers of Ω as indicated above; if, however, we neglect the realm of rationality Ω and permit the discussion to extend to the general realm of all algebraic numbers, there is something which corresponds to the greatest common divisor in the theory of natural numbers. There is then, however, no such thing as a prime number and the theorem regarding the unique factorization of a number into its prime factors does not exist. However, we can not lose sight of the theorem regarding the unique distribution of a number into its prime factors and therefore the conception of the theorem regarding the unique distribution of a number into its prime factors and on this account the conception of the divisibility through the complex $[\alpha_1\beta]$ is no longer superfluous; it becomes necessary.

The above definition is applicable to the complex $m = [\alpha_1, \alpha_2, \alpha_3, \ldots]$, which consists of more than two algebraic numbers in Ω. Accordingly, an algebraic number λ is said to be divisible by this complex if $\lambda = \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots$, where the ξ's are integers in Ω.

The investigation may be restricted in that the ξ's are required to be rational integers. The collectivity of all algebraic numbers that are expressible through the linear form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots, x_1, x_2, \ldots$ being rational integers was called by Dedekind a *modul*. It may be observed, if functions of one or more variables with coefficients that belong to a fixed realm are written in the place of the α's, that the modular systems of Kronecker are nothing other than the moduls of Dedekind.

A number λ is said to be divisible by a modul or modular system if λ is a number of the modul, that is, if λ is contained in the modul, and that is, if rational integral numbers $x_1, x_2, \ldots$ may be found such that $\lambda = \alpha_1 x_1 + \alpha_2 x_2 + \ldots$. Here we have encountered something which at first may appear as a "confusion of language" in that the conception of "divisibility" and of being "contained in" which heretofore have been opposed are now identical.

Dedekind offers also the following definition of a modul in order to give the theory a more philosophic setting in that it is independent of the notion of linear forms: *A modul is a system of numbers such that the difference of any two numbers of the system is a number of the system.* If $a_1, a_2, \ldots a_n$ are a finite number of quantities of Ω and if there is no linear relation among them with rational coefficients, these quantities constitute a *basis* of a *finite* modul. In this case $[\alpha_1, \alpha_2, \ldots \alpha_n]$ is called a modul of the nth order. We are then led through easy steps to the conception of equality of moduls, the greatest common divisor, the least common multiple, the product and the quotient of moduls.

All numbers of the modul **m** are said to be divisible by **m**, and if α is any such number, then is $\alpha \equiv 0$ (mod **m**); and α is said to be congruent to β (mod **m**) and written $\alpha \equiv \beta$ (mod **m**), if $\alpha - \beta$ is divisible by **m** and that is, $\alpha - \beta$ is a number of **m**.

If **a** and **b** are two moduls and if **b** is divisible by **a**, then the numbers of **a** fall with respect to **b** into a certain number of classes, which number is denoted by (**a**, **b**). Any number of a class may be selected as a representative of the class. We thus have as many representatives as there are classes.

These representatives have the following characteristics:

(1) They are all divisible by **a**;

(2) The difference of no two of them is divisible by **b**;

(3) Every number that is divisible by **a** is congruent to one of these numbers (mod **b**) and to only one.

It is evident that if **a** is divisible by **b** that (**a**, **b**) = 1.

In general without assuming that **b** is divisible by **a**, make the assumption that n elements $\beta_1, \beta_2, \ldots \beta_n$ of **b** are such that

$\beta_r = e_{r1}\,\alpha_1 + e_{r2}\,\alpha_2 + \ldots + e_{rn}\,\alpha_n$ ($r = 1, 2, \ldots n$), where the e's are rational numbers. Denote the determinant of these expressions by C. It may be proved that

$$\frac{(\mathbf{a}, \mathbf{b})}{(\mathbf{b}, \mathbf{a})} = \left| \, C \, \right|.$$

If in this expression **b** is divisible by **a**, then the e's are integers and (**a**, **b**) = | C |.

By definition (1) a finite modul **a** is said to be *algebraic*, if all the numbers that are divisible by **a** are algebraic. (2) An algebraic modul **a** is an *integral algebraic* modul if all the quantities that are divisible by **a** are algebraic integers. (3) An integral algebraic modul is a *unit modul*, if 1 is divisible by this modul. These three definitions are restricted to finite moduls. To show that a finite modul is algebraic, it is only necessary to show that the modul has a basis which consists of only algebraic integers.

Theorem.—If **a** *is a finite modul which belongs to an algebraic realm* Ω*, there exists a finite modul* **b** *of* Ω *such that* **ab** *is a unit modul which consists only of algebraic integers.*

Theorem.—All the algebraic integers of a realm of the nth degree constitute a finite modul of the nth order.

There exists always in such a realm a modul **v** whose discriminant has a minimum value. This modul **v** is such that $\mathbf{v}^2 = \mathbf{v}$ and $\frac{\mathbf{v}}{\mathbf{v}} = \mathbf{v}$, so that **v** plays the same rôle in the algebraic realm as 1 does in the usual realm of rational numbers.

Theorem.—If **a** *is an arbitrary modul of the nth order in a realm* Ω *of the nth degree, then every number* β *of* Ω *may through multiplication by a rational integer be transformed into a number that is divisible by* **a**.

An *ideal* is a modul of the nth order in a realm of the nth degree formed of the complex of values of a linear form $\alpha\xi + \beta\eta + \gamma\zeta + \ldots$, in which $\alpha, \beta, \gamma, \ldots$ are integral or fractional algebraic numbers of a fixed realm Ω and where $\xi, \eta, \zeta, \ldots$ are any integers of Ω. If **a** is an ideal and **v** the modul defined above, then is **va** = **a**. This is characteristic of an ideal and serves to define it, being in fact the best definition of an ideal. It may be proved that the greatest common divisor, the least common multiple, the product and the quotient of two or more ideals are ideals.

If α is an arbitrary number of Ω, then is $\mathbf{v}\alpha$ an ideal, a *principal* ideal. *Theorem.—If* **a** *and* **b** *are two ideals of* Ω*, there is one and only one ideal* **k** *such that* **ak** = **b**.

It may be proved that

$$\frac{(\mathbf{v}, \mathbf{v}\eta)}{(\mathbf{v}\eta, \mathbf{v})} = N\,(\eta) = N\,(\mathbf{v}\eta),$$

where η is an arbitrary number of Ω and N denotes its norm. This property of the principal ideal $\mathbf{v}\eta$ leads to the following definition. If **a** is an arbitrary ideal, then is

$$N(\mathbf{a}) = \frac{(\mathbf{v}, \mathbf{a})}{(\mathbf{a}, \mathbf{v})}$$

from which it follows that N(**v**) = 1.

An ideal is said to be integral if it consists only of algebraic integers and this is true if $\alpha, \beta, \gamma, \cdots$ above are integers. We may observe further that in every such ideal there exist n integers $\gamma_1, \gamma_2, \ldots \gamma_n$ which have the property that every integer ι of the ideal may be expressed in the form $\iota = g_1\,\gamma_1 + g_2\,\gamma_2 + g_3\gamma_3 + \ldots$; where the g's are rational integers.

If **g** is an integral ideal, observe that

$$N(\mathbf{g}) = \frac{(\mathbf{v}, \mathbf{g})}{(\mathbf{g}, \mathbf{v})} = (\mathbf{v}, \mathbf{g}).$$

And this, as seen above, is the number of classes into which the integers of Ω may be distributed with respect to the integral ideal **g**. This number plays fundamental rôles in the further development of the theory, for example, in the proof of Fermat's Lesser Theorem for ideals, in the determination of the number of classes into which an integral ideal may be distributed, etc.

Theorem I.—If two integral ideals **a** *and* **b** *are relatively prime, and that is, have no ideal factor in* *common save* **v**, *and if* **c** *is a third ideal, then if* **bc** *is divisible by* **a**, *the ideal* **c** *is divisible by* **a**.

Theorem II.—If **a** *and* **b** *are two integral ideals that are relatively prime and if* **a** *and* **c** *are two integral ideals that are relatively prime, then the greatest common divisor of* **a** *and* **c**, *this common divisor of* **a** *and* **c**, *this common divisor being* **v**.

Theorem III.—An integral ideal **a** *is divisible by only a finite number of other ideals.*

If an integral ideal **a** has the property that it is divisible by itself and by no other ideal save **v**, it is called a *prime* ideal.

Theorem IV.—If a product of several integral ideals is divisible by a prime ideal, one of the ideals is divisible by this prime ideal.

Theorem V.—The Fundamental Theorem. Every integral ideal that is not **v** or a prime ideal may be *factored into a product of prime ideals and this factorization is unique.*

Observe that every algebraic integer when multiplied by **v** is a principal ideal and that the above theorems are applicable to it.

The integral ideals constitute one branch of the general theory of moduls. This general theory in its incipience comprises the Kronecker modular systems and indeed many other branches of mathematics that emanate from the general realms of rationality and include the Minkowski geometry of numbers, the treatment of the moduli of periodicity of the Abelian Integrals, etc.

As a rule the text-books on the usual theory of numbers make the positive integer the starting point and the theorems regarding such integers form the foundation of the theory; it appears also that the text-books on the theory of algebraic numbers are going to start with the integral ideal. It should be emphasized that such ideals have their general setting in the general modul theory just as *number* is the more general concept of the usual positive integer.

The theory as outlined above may be made dependent upon the fundamental theorems of Dedekind as given by him in the "Begrundung der Idealtheorie. Göttingen-Nachrichten 1895."

I. *If the ideal* **c** *is divisible by the ideal* **a**, *there exists an ideal* **b** *such that* **c** = **ab**.

II. *Every ideal may be changed through multiplication by a properly chosen ideal into a principal ideal.*

III. *Every finite modul that is different from zero may through multiplication by a properly chosen modul be changed into a modul which contains the number 1 and further consists of only integers.*

IV. *If* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *denote any n numbers that are not all zero of a realm* Ω *of the nth degree, it is possible to derive by rational operations n other numbers* $\beta_1, \beta_2, \cdots \beta_n$ *of* Ω *which satisfy the two conditions, first that* $\alpha_1 \beta_1 + \ldots + \alpha_n \beta_n = 1$, *and secondly, that the* n^2 *products* a_r, β_s *are all integers.*

If any three of the above theorems are proved, the fourth follows as a consequence.

Observe that throughout the entire discussion of this article a fixed stock-realm R has been the realm of reference. This stock-realm was the usual realm of rational numbers. The theorems derived have been for the numbers of another realm, say Ω_1, which was deduced by adding (adjoining) to R an algebraic quantity. This algebraic quantity was in turn the root of an algebraic equation whose coefficients were rational numbers (and that is, numbers of R). It is possible to introduce a third realm Ω_2 which bears towards Ω_1, the same relation as Ω_1 had with respect to R, and so on indefinitely. Instead of the ideals that are introduced for Ω_1 other (more general) ideals exist for Ω_2 through the introduction of more general norms, discriminants, etc. It may be proved that the same rules, laws and principles exist in the more general realms as were true in Ω_1. And thus it becomes manifest that the principles of arithmetic are true universally and that is, in any algebraic realm whatever taken with respect to any arbitrary algebraic stock realm as realm of reference.

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SCIENTIFIC EVENTS

THE NANSEN POLAR EXPEDITION

THE Christiania correspondent of the London *Times* writes: The news that Dr. Nansen, after nearly 30 years spent in labor far away from the Arctic, will again return to the work of his youth is sure to attract general attention. Dr. Nansen has taken his decision. He will not only join the North Pole Expedition of the German Commander Bruns, but he will become its leader. By his famous expedition with the *Fram* in the years 1893–96, Dr. Nansen gained a reputation which, coming after his crossing of Greenland, placed him in the highest rank of Arctic explorers, and his interest in the North Polar basin has not waned.

Just as Dr. Nansen in 1893–96 had no ambition of reaching the North Pole apart from scientific exploration, so he is without this ambition on this occasion also. At the meeting of the Geographical Society he declared the flight over the Pole to be a matter of secondary importance, and in a subsequent interview he expressed the hope that Captain Amundsen will reach that goal next summer. Dr. Nansen will certainly not try to overtake Amundsen.

The projected Nansen Expedition is primarily in-