been brilliant contributors to knowledge, although their general manner of living may have been an injury rather than a service to mankind. We need to be grateful for the constructive service of each life, and our criticism of those who have passed and of those who are still active needs to have a broad friendliness as its basis. I believe, too, that a good scientist should be a good Christian, and a good Christian should be a good scientist in his method and work, as both are seeking the truth and the fundamental principles underlying their respective fields of endeavor.

Besides the necessity for each individual to train and conquer himself and to exercise such influence as may be possible on those within his immediate environment, there is great need for him to engage in cooperative public work, by associating with others of similar aspirations, and bringing legitimate influence to bear on all agencies that are concerned in any way with the educational system of the people, from the kindergarten to the university, from the leaflet of the advertising promoter to the great newspapers, magazines and books that make up the thousand and one publications of our day. His influence must also be brought to bear upon the important visual agencies of the motion-picture screen and every other form of illustration, as well as on all those agencies that are seeking to develop "the consciences, the ideals and the aspirations of mankind." The scientific method must be applied to all these factors if we have faith in its ideals.

Is it not practicable for the association to organize a progressive, live committee of men and women to deal with the popularizing of scientific knowledge? It might arrange special sessions for the public to which the layman could go with the feeling that they were for his entertainment and his instruction and not solely to arouse the interest of specialists in their particular field of research. Of all human beings, the child is the greatest and most active investigator of all that pertains to his environment. Why not provide for a junior section of the American Association, and last and in some respects the most important, a woman's section and sessions, at which all the scientific problems of peculiar interest to woman could be considered? We have a strong nucleus of women members, but they should be one of the great influences within the association for developing and carrying forward its work. Then there is the much discussed business man, who has a more or less hazy conception of science and scientific method, depending on whether he considers it affects his interest for good or evil. He would be a better business man, a better citizen and more successful in all his relations in life if he had a working knowledge of scientific method and principles at his command.

Every member of our association should work individually and collectively according to his or her opportunity and ability in supporting the scientific method and in insisting that, in all education of every kind and degree and for all classes, the purpose is to develop without prejudice or preconception of any kind a knowledge of the facts, the laws and the processes of nature in all natural and human relations. The natural weakness and incompleteness of all things of human origin will frequently baffle, mislead and confuse, and may even apparently bring temporary defeat, but in the long run there is no other way to eradicate sciosophy, advance the physical, mental and moral welfare of the race and justify our existence and opportunities for service as sentient human beings.

The Pilgrim fathers knew little of science, but they brought the great principles of law, truth, freedom and faith in God to America. Are we doing all in our power to perpetuate and develop them in connection with the multiplex activities of the world of to-day?

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THE FOUNDATIONS OF THE THEORY OF ALGEBRAIC NUMBERS[1]

ONE approaches with considerable hesitation a task in which on the one hand he is to give to the general reader some notion of an abstract scientific subject, while on the other hand he is to write something worth reading by the specialist in this subject. The task before me is to write something that is fit reading for the non-mathematician which is fit for the mathematician to read. My thesis has to do with the foundations of a theory which is of much interest to scientists and to mathematicians. I shall try to present it in such a way that one may foresee the methods that are adopted in the development and extension of this particular mathematical subject and may observe the generalizations that as a rule are applicable to mathematical theories. I shall attempt further to show that the truths established in mathematics must be used to standardize the truths, if such exist, of the general laws and principles that are derived for other sciences, and, in a measure, for all branches of human knowledge. Emphasis is also made of the importance of the universality of the truths that are found in the study of mathematics. A simple example

[1] Address of the vice-president and chairman of Section A—Mathematics, American Association for the Advancement of Science, Washington, December 31, 1924.

will illustrate. It is said that Newton on seeing an apple fall to the ground discovered thereby the law of gravitation. This law, which is applicable to all bodies on the earth, was held by Newton, through a process of observation and induction, to be true for all the heavenly bodies; that is, he held that the same laws which are true for the apple are true for the earth, the system of planets and the fixed stars, however distant. We shall see later that the entire theory must be modified. And it will be seen that in the testing of the accuracy and generality of the Newtonian laws recourse must be had to the very theory that I propose to outline in the present paper.

It may be observed that twenty-five years ago we were led to believe that certain theories of physics were firmly established upon accurate and sure bases; that advances were to be made upon the firm structure already established; and that further developments for the most part were to be had only through experimentation with elaborate and costly apparatus. Since then the entire structure of the science has been greatly altered; many marked and revolutionary changes have been made; the older text-books have been discarded. And this is what has been experienced in the oldest and most accurate of the experimental sciences!

With such unstable conditions in the sciences in which the theories and laws are created through experimentation, observation and induction, and with our ever-changing philosophies which underlie the very roots of civilization, where, may we ask, are we to seek the truth and, in fact, what is the truth? It is fortunate that our mathematical theories offer an immediate answer in that the experimental sciences and the basic philosophical theories may be accepted as fixedly established and may be gauged as accurate in so far as they have definite mathematical bases or conform with those logical principles which are essentially mathematical. It is, therefore, of the utmost importance in any branch of mathematics to establish for it universal truths, truths that however generalized remain invariable.

It is worthy of note that the mathematical theories in their generality have usually preceded the other sciences; in many cases the practical sciences have suggested fundamental mathematical principles. These principles, in turn, when developed have become important branches of mathematics, the advances usually being much more rapid than in the experimental sciences. Thus the practical value of a mathematical theory may not become evident until long after the theory has been developed. There are many branches of mathematics which, seemingly evolved solely for their own sake and apparently as abstract exercises for mathematicians, have eventually been of the utmost practical value. It follows that we can not consider mathematics apart from their utilitarian value, or along with it, but rather we must consider the overlapping of the two. In a great measure, it is due to the fact that abstract mathematics is a thing in itself that its progress has been so rapid. The above remarks are illustrated in the subject of this paper, namely, *the theory of algebraic numbers*, or, *general arithmetic*, the meaning of which becomes manifest as the discussion proceeds.

It has been said that "arithmetic stands alone in the simplicity of its fundamental postulates, the accuracy of its concepts, the purity of its truths." *General arithmetic* has burst from the bonds in which of old it was confined. The ancients would restrict it within the confines of geometry. The theory is a modern creation. However, it wants in nothing that made the older arithmetic attractive. Starting with the conception of the absolute whole number, the domain of arithmetic has been enlarged step by step. These advances, like most other advances that have had to do with the extension of human knowledge, have always been attended with slowness, doubt, misgivings and adverse criticism. It is seen that fractional numbers were added to integers; negative numbers to positive numbers; imaginary numbers to real numbers. The early algebraist called the negative roots of an equation *false* roots. There are numerous problems where the quantities sought from an equation do not admit *opposites;* and there are innumerable cases where only positive whole numbers have a meaning, fractional quantities having no significance. However, this should not, as it did, argue such quantities into the background, when it comes to the complete solution of the algebraic equations. For, on the other hand, the reality of fractional or negative numbers is sufficiently warranted in that in innumerable cases they find adequate justification.

Similarly, the imaginary quantities were accepted with much hesitation and mistrust. The metaphysics of these positive and negative real integers, as well as of the positive and negative complex numbers, is susceptible of visualization if concepts of a geometrical nature were desired. The absolute or positive integers may be represented by a series of points that are situated on a straight line at equal (which we call *unit*) distances apart. A fixed point is taken as the origin and marked zero, the next point on the line being 1, the point after this being 2, etc., the line extending in this (positive) direction indefinitely. For a presentation of the negative numbers, it is only necessary to extend the line on the opposite side of the origin, the unit negative distances being marked on it beginning with the origin.

To visualize the complex integers suppose the above line lies in a plane that is unlimited in every direction. Parallel with this line and at unit distances apart, draw other lines. Then through the origin draw a line perpendicular to this system of lines. The points of intersection on the one side of the original line with the system of lines just indicated may be denoted by 1i, 2i, 3i, . . . , while those in the opposite direction (on the other side) of the original line, may be called —1i, —2i, . . . Parallel to this latter line, draw a system of lines that are everywhere unit distances apart. This system of parallel lines makes with the first system a distribution of the entire plane into small squares. The vertices of these small squares correspond to complex integers. Observe, however, that the second system of lines has on the plane as much of an objective reality as has the first system. And, in fact, in the plane these so-called imaginary points (or quantities) have as *real* a significance as do the real points, positive and negative, on the line first drawn. Thus it is seen that to the points of intersection of these two systems of parallel lines there correspond complex integers and *vice versa;* and, observe that the points on the line first drawn, that is, the real integers, are included in the general class of complex integers.

The question now is: Do these complex integers obey the same laws and rules that have been established for the real integers? In particular, if a complex integer be resolved into factors, which can not be further resolved into factors, is the final result unique? And that is, can the distribution of a complex integer, as is the case of a real integer, into its prime factors be effected in only one way, if the order in which the prime factors are arranged is neglected? To all these questions there is an affirmative answer. Note, however, that we have no right to assume *a priori* that the theorems true for a restricted class of integers which lie on a line are also true of the more comprehensive class of integers which as above defined lie on a plane.

Although the bounds of the ancient geometricians do not allow us to go much further, we shall not stop with the introduction of the so-called complex numbers. Instead, we shall extend the limits of our investigation by the introduction of more general types of integers and numbers and, correspondingly, we shall introduce more extended geometries. Those who had to do with the introduction of these so-called *algebraic numbers* and who solved the questions just raised were the ones who established the theory with which we are concerned. Gauss, a mathematical scientist, algebraist and arithmetician of very great ability, extolled Fermat, Euler, Lagrange and Legendre as men of "matchless honor in that they had opened the way to the sanctuary of this divine science (theory of numbers) and had revealed how great were the treasures with which it was enriched." These men, who were the forerunners of the particular kind of numbers (algebraic) that are treated here, are next introduced.

Cajori, in his valuable "History of Mathematics," tells us that Diophantus, who flourished about 250 A. D., at Alexandria, completely divorced algebra from the methods of geometry. Equations which admit as solutions positive numbers when there are more variables than equations, are known as *Diophantine* equations. The celebrated cattle problem of Archimedes (stated in Cajori's history), and which antedates this period about five hundred years, is a typical example. Simpler examples are introduced here illustrative of problems whose answers are necessarily positive integers:

Problem 1°: A farmer wished to stock his farm with exactly one hundred animals. He paid 50 cents a head for pigs, three dollars for each sheep and ten dollars apiece for cows. He spent exactly one hundred dollars. How many of each did he buy?

Problem 2°: A lady told the postmaster to give her exactly one dollar's worth of stamps and in such a way that there are as many one cent stamps as two cent stamps and the remainder only in three cent stamps. How long did she wait for the stamps?

Problem 3°: A certain Egyptian queen, meeting her three suitors, gave from her basket one half of the cherries and one over to one of the suitors, one half of the remainder and one over to the second, and one half of what was left to the third suitor. She then agreed to accept that suitor who could first tell her how many cherries there were in the basket.

The answers to this problem are 14, 22, 30, 38, 46, . . . The second problem admits no answer and the first problem has one answer.

Fermat (1601–1665), a lawyer and mathematician of exceptional ability, interested himself in restoring and translating into Latin certain books and fragments that had been handed down from the ancient mathematicians, in particular, those of Apollonius of Perga and of Diophantus. Parenthetically it may be observed, if one will read Fermat's "Oeuvres" (Tannery Edition) III, p. 121, and then the works of Isaac Barrow (1630–1677), who was versed in Fermat's works, that Barrow's pupil, Isaac Newton, did not have far to go in the discovery of the differential calculus.

In his "Observations sur Diophantus," *loc. cit.*, p. 241, commenting upon the solution of the equation $x^2 + y^2 = z^2$, which was first completely solved by Diophantus, Fermat writes:

Au contraire, il est impossible de partager soit un cube en deux cubes, soit un bicarré en deux bicarrés, soit en général une puissance quelconque supérieure au carré en deux puissances de même degré; j'en ai découvert une démonstration véritablement merveilleuse que cette marge est trop étroite pour contenir.

The proof of the impossibility of the solution of the equation $x^n + y^n = z^n$, $(n > 2)$ in integers, is the celebrated Fermat greater theorem. A large number of problems of a similar nature due to Fermat are found among others in the "L'Inventum Novum" (published in Fermat's works) of Father Jacques de Billy, who, together with his intimate friend, Claude-Jasper Bachet, as well as Euler (1707–1783) and Lagrange (1736–1813), interested themselves particularly with the solutions of Diophantine equations. Such problems led at once to the study of the theory of indeterminate equations, congruences, etc. In particular, Legendre (1752–1833) gave a proof of the so-called law of quadratic reciprocity (see Crelle's *Journal,* Bd. 30, p. 217).

Gauss (1777–1855), in the thirtieth section of his "Theoria Residuorum Biquadraticorum," writes:

The theorems relative to the biquadratic residues are not shown in their greatest simplicity and in their genuine beauty until the field of arithmetic (Campus Arithmeticae) is extended to imaginary quantities and in such a way that numbers of the form $a + ib$ (i denoting $\sqrt{V-1}$, while a and b denote indefinitely all real integers from $+\infty$ to $-\infty$) constitute without restriction the object of this field. These numbers we shall call complex integers and in a manner that the real integers are not opposed to the imaginary but are to be regarded as constituting a class of integers among the imaginary integers.

It must not be assumed for a moment that Gauss was the first to introduce imaginary quantities into analysis. Lagrange, for example, seemed as familiar with them as we are to-day.

Abel (1802–1829) (see his Works (Lie edition) II, p. 219), writes: "What do we understand by the algebraic solution of an algebraic equation." This question he answers as follows:

In general there are two different cases according as the coefficients will be *rational functions* of a certain number of quantities x, z, z′, z″, . . . which will contain at least one independent variable. . . . In the second case where we consider the coefficients as constant quantities, we may regard the coefficients formed from other constant quantities by the aid of rational operations.

In other words, when we consider algebraic equations, their coefficients belong to certain fixed fields, domains or *realms of rationality* in which are allowable the rational operations of arithmetic. And, if any new quantity whatever is added or adjoined to this domain, the quantities in the resulting new domain consist of all rational combinations of this new quantity together with the quantities that at first constituted the domain. Our calculation, computation or process of reasoning which has to do with the solution of the algebraic equations has to be done with reference to a domain regarded as fixed in the premises, which domain we shall call a *stock-realm.* As we ourselves are rational when we think, and as this process of thinking is carried on with reference to a domain in which the admissible operations have already been designated as rational, I prefer to designate such domains as *realms of rationality,* a terminology which was adopted by Kronecker (1823–1891). For example, if to the natural realm, in which all the usual computations are made, we adjoin the quantity i, we have a realm consisting of all the complex numbers (fractional and integral) as defined above. If further we adjoin a quantity u, we obtain a realm consisting of all rational functions of u whose coefficients are complex quantities. This process may be continued indefinitely. We may thus form realms that are extended without limit.

With Abel (Works I, p. 479), we may say that a function $\varphi(x)$ whose coefficients are rational functions of a certain number of known quantities a, b, c, . . . is called *irreducible,* when it is impossible to express any of its roots through an equation of lower degree whose coefficients are likewise rational functions of a, b, c, . . . However, by adding (adjoining) other quantities to the fixed realm, the given function may well be factorable in the extended realm. And this notion is the one given and employed by Galois (1811–1832).

In the natural realm, *i.e.,* the realm of numbers usually called rational, an integer, say 6, satisfies an equation $x - 6 = 0$. The quantity i satisfies the equation $x^2 + 1 = 0$. In general an algebraic number satisfies an equation of the form $a_\sigma x^n + a_1 x^{n-1} + \ldots + a_{n-1} + a_n = 0$, where the a's are quantities that belong to a fixed realm. If $a_\sigma = 1$, and the other a's are integral, these algebraic numbers are called algebraic integers; and if further, $a_n = 1$, these algebraic integers are called algebraic units. These units play a rôle analogous to the one of ± 1 in the usual realm.

If to the natural realm any number of algebraic quantities be adjoined, the resulting realm is called an *algebraic* realm; and any quantity of this realm, and that is, any rational combination of the numbers that have been adjoined together with the original numbers of the realm, is itself an algebraic number and satisfies an algebraic equation. It may be proved that any algebraic number multiplied by a suitable rational integer is an algebraic integer in the sense defined above. These integers associated with their respective realms are the subject of our present inquiry.

The theory of algebraic numbers and the Galois theory of equations have their common origin in the general theory of algebraic realms, so that these realms constitute an essential rôle in questions of algebra and analysis. In fact the focus of the modern theory of numbers is the theory of the algebraic realms from which radiate its relations with algebra, functions, geometry and analysis. It is seen that the definitions of Dedekind, the arithmetical concepts of Weierstrass and Cantor's general notion of number lead to an arithmetizing of the function theory which finds exposition in the principle that a theorem may be regarded as proved only when in the final analysis it may be reduced to the relations which exist for rational integers. This notion was carried to the extreme by Kronecker, who often declared, "Die ganze Zahl schuf der liebe Gott; alles Uebrige ist Menschen Werk." By the introduction of number into geometry, the modern non-Euclidean geometry may be developed from a purely logical basis; while the so-called geometry of numbers (Minkowski) serves to illustrate and oft-time to derive very simply general arithmetical theorems by means of concepts considered geometrical.

In the advancement of science there is a certain continuity. New methods are employed in the solution of old problems; and when the older problems are the better understood and further developed, new problems of themselves are proposed, and so on indefinitely. In mathematics a simple example will illustrate: The Riemann surface offered a means of studying many-valued functions in $z = x + iy$. Through these surfaces a new kind of geometry was revealed for the inversion of Abelian integrals; the hyper-geometric series could be regarded from a different viewpoint; automorphic functions came into being; while the elliptic modular functions took on a new interest.

The geometry of Euclid has been more or less restricted in that certain axioms must have their being. Do away with the one that pertains to parallel lines and we are in a more general geometry, which of course includes the Euclidean geometry. This new geometry may be regarded as a particular kind of number triply extended in which the square of an element of arc is expressed through a quadratic form of the elements of the coordinates. Further, it is possible to introduce, instead of a triple multiplicity, a quadruple multiplicity by the introduction of a fourth variable t in a space-time geometry about a space-time origin $x = 0 = z = t$. And this more general geometry includes the two just mentioned. In this extended geometric realm we have among other things to investigate what are the analogues of the Newtonian mechanics and other phenomena included in the usual physics. The notions of continuity, maxima and minima, extremals, etc., are first of all to be considered and definitely fixed. These extended geometric realms may well be called "bodies" (Körper). Thus we have a geometric realm, a "body," in which geometric, physical, electrophysical phenomena have their being and as such are to be studied just as in the number theory exist extended realms in which generalized number is to be studied. This latter is the problem before us, a problem which in a more or less degree has been solved. Thus, as we said in the beginning, mathematics must blaze the way through the forest of uncertainty and let the physical sciences follow the trail.

It would take us too far afield to introduce here the so-called "Welt-postulat" of Minkowski. This was founded upon three very simple assumptions due to which the motion of a material point is made to depend upon a system of four differential equations. These reduce to the three Newtonian equations when the velocity of light is made indefinitely large, or when for the time t the local time (Eigenzeit) τ of the material point is substituted, the fourth equation in this case offering an expression for the law of conservation of energy; or inversely, if we start with the postulate of relativity in the development of mechanics, the equations of motion follow at once from the energy theorem. The Newtonian theory of gravitation must be modified to satisfy this "Welt-postulat" and in the doing of it the Newtonian mechanics are made to harmonize with the more modern electrodynamic theory. It would thus appear that the real meanings of the laws of nature are to be properly understood when the elements that constitute the generalized domains (Körper) are correctly interpreted, and when the laws which the quantities in these realms must obey are definitely determined and fixed.

Legendre's quadratic law of reciprocity was mentioned above and it was seen that the theorems relative to this theory were best understood when the usual realm of arithmetic is widened by the adjunction of the quantity i, the realm thus becoming R (i). So also the meaning of the cubic law of reciprocity is revealed in its clearness when the well-known quantity ω is adjoined to the usual realm of arithmetic. In these two extended realms $R(i)$ and $R(\omega)$ there exist no essentially different principles from those of the usual arithmetic. For example, by definition the number $a + bi$ is integral when a and b are rational integers. Observe that this number satisfies the equation $x^2 - 2ax + a^2 + b^2 = 0$, where the coefficient of x^2 is unity and the other coefficients are rational integers. The sum, difference and product of any two such integers is an integer. If further α, β, γ are any three such integers and if $\alpha = \beta\gamma$, we may say that α is divisible by β or by γ; β is a divisor of α, so also is γ. Further, every complex integer is factorable into prime factors in only *one* way. A complex integer $a + bi$ may be taken as a modulus of linear, quadratic and higher congruences with associated theorems that are analogous to those for real integers and moduli. Corresponding to every integer $\alpha = a + ib$ there exists a system of $a^2 + b^2$ integers such that no two are congruent (mod α), whereas every integer of $R(i)$ is congruent to one of these integers (mod α). Euclid's method of finding the greatest common divisor is also applicable. The so-called Fermat Lesser Theorem and the Wilson Theorem may be applied to all these integers, and there exist the analogous theorems of primitive roots and the analogous laws of quadratic and cubic reciprocity. The principles just enunciated constitute what is known as the *Arithmetic* of these realms. For the more general realms there exist additional theories, particularly one which treats of the numbers of classes into which the units of a realm may be distributed and another which has to do with the determination of the number of fundamental units which are found in a given realm. These two theories of Dirichlet are discussed at greater length later.

(*To be concluded*)

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THE ECLIPSE OF JANUARY 24, 1925

THE eclipse of the sun taking place on January 24, 1925, is chiefly remarkable by the fact that the zone of totality passes over an area more thickly populated than any other of modern times. At sunrise somewhat west of Duluth the zone includes such cities as Toronto, Buffalo, Rochester, Hartford and New Haven, while close to the northern limit are Syracuse, Springfield, Providence and to the southern limit Wilkes-Barre and New York.

It seemed advisable to make use of this fact partly to secure an exact delimitation of this zone and partly to interest the people generally in the scientific and spectacular features of the phenomena. A special committee of the American Astronomical Society was therefore formed to take charge of the general publicity work required.

The most important factor was of course to secure the cooperation of the daily press. Articles giving details free from technical terms have been sent out through the usual distributing agencies to all newspapers whether published in the big cities or in the small country towns. Starting early in November, these or extracts from them were very generally printed and by the end of the month, public interest was becoming very evident.

Newspapers with large resources arranged for interviews and special feature "stories," while other articles were contributed by astronomers, professional and amateur, for the local press in the smaller cities. Articles in the magazines were left to the enterprise of their editors.

The delimitation of the northern and southern boundaries of the zone is being made by asking for answers from any one interested to certain questions framed in such a way that a person without a knowledge of astronomy or of eclipse phenomena can give information as to whether the eclipse was total or not at his position. Observations of the edge of the moon's shadow as it passes over the earth, of the visibility of the corona and of the sun itself are requested. Just within the zone, the time of duration of totality also furnishes a good position. In connection with this, professional astronomers have been asked to make as many observations of the moon during the month preceding and that following the eclipse, so as to secure a knowledge of its path during this period which shall have the highest possible accuracy. The main reason for this arises from the fact that only during a solar eclipse can an accurate position of the moon be obtained when it is close to the sun. This therefore furnishes positions through a lunation including one at new moon, which last at other times is missing.

The comparison of the different classes of observation, meridian, extra-meridian, photographic and by occultations will be a by-product of the campaign.

The eclipse is also unique in the fact that the zone of totality includes some ten or twelve observatories. These of course have their programs for observation, and other observatories are as usual sending expeditions into the zone. A general feature of most of their work will be the photography of the corona. The time of the year, however, makes the weather problem doubtful, the best chance for clear skies being near the coast and that some fifty per cent. It may happen, however, that the sun will only be free from clouds in parts of the zone not occupied by the professional astronomer and here again the cooperation of the public is asked. Any photograph showing the corona sufficiently to indicate its type is better than none, whatever the scale. It is also suggested that