SCIENCE

A Weekly Journal devoted to the Advancement of Science, publishing the official notices and proceedings of the American Association for the Advancement of Science, edited by J. McKeen Cattell and published every Friday by

THE SCIENCE PRESS

11 Liberty St., Utica, N. Y. Garrison, N. Y.

New York City: Grand Central Terminal

Single Copies, 15 Cts. Annual Subscription, $6.00

Entered as second-class matter January 21, 1922, at the Post Office at Utica, N. Y., under the Act of March 3, 1879.

VOL. LV FEBRUARY 24, 1922 No. 1417

A MECHANICAL ANALOGY IN THE THEORY OF EQUATIONS[1]

To the mathematician the solution of a problem is the more interesting if it utilizes methods and principles from fields that at first glance seem foreign to the one in which the problem lies. The question of whether a linear differential equation has algebraic solutions is sufficiently important to attract attention of itself, but its answer by reference to the properties of regular polyhedrons has become a mathematical classic. Such analogies are not, however, to be regarded as mere *tours de force* whose purpose is only to astonish, or to appeal to a certain esthetic sense; the instance just mentioned shows that the new point of view may disclose wide vistas hitherto undiscerned. If there is a choice of terms in which the analogy may be stated, the formulation which is most concrete and most striking may also be the most illuminating.

Such considerations as these, doubtless, have led to the description of what are essentially vector methods with complex variables in terms of mechanical systems. I propose here to discuss the progress that has been made by the aid of such an interpretation in studying the distribution in the complex plane of the roots of algebraic equations in one variable.

On the algebraic side the chief purpose of the investigations to be considered has been to obtain what may be called *theorems of separation, i. e.,* theorems which state whether roots of an equation do or do not lie in specified regions of the complex plane. Such theorems may also state how many roots lie in the specified regions, or may give limits, inferior or superior, for the number of roots thus situated. These regions may be defined in terms

[1] Address of the vice-president and chairman of Section A—Mathematics, American Association for the Advancement of Science, Toronto, 1921.

of the roots of other polynomials; we are then concerned with *relative distributions* of the roots of two or more polynomials.

Theorems of separation for real roots of real equations are numerous, and are among the most familiar results in elementary mathematics. I need only mention Descartes' rule, which gives a superior limit for the number of roots on the positive real axis, or Sturm's method for obtaining the exact number in any real interval. Rolle's theorem, in the form which states that between each consecutive pair of real roots of a real polynomial $f(x)$ there lies an odd number of real roots of the derived function $f'(x)$, is perhaps the most important proposition concerning relative distributions of real roots of two real polynomials.

No such progress has been made with similar propositions for complex roots, although the widening of the field of observation from the real axis to the complex plane vastly increases the range of possibilities. To be sure, we have extensions of Sturm's theorem, and other methods, both algebraic and transcendental, which give criteria for the exact number of roots within a region, but in practice these prove so cumbersome as to be of little use. The great desideratum is a body of results whose simplicity and range of applications would make them comparable with Rolle's theorem, or the Budan-Fourier theorem in the real case. As Jensen has remarked, the solution of important problems regarding the zeroes of transcendental functions may be dependent upon progress in this direction.

The significance of Rolle's theorem naturally led to attempts to extend it to the complex plane almost as soon as the now familiar geometric representation of complex numbers had been adopted. A line of attack is clearly indicated by the identity of the logarithmic derivative

$$\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \dots + \frac{1}{x - \alpha_n},$$

where $f(x)$ is a polynomial of degree n, whose roots are $\alpha_1, \alpha_2, \dots, \alpha_n$, and $f'(x)$ is the first derivative of $f(x)$. Gauss was probably the first to give this a mechanical interpretation which depends on the representation of a complex number $x - \alpha$ as a free vector whose length, $|x - \alpha|$, and direction are those of the directed line segment from the point which corresponds to α, or, more briefly, from the point α, to the point x. The conjugate of the reciprocal of $x - \alpha$, which may be denoted by the symbol $K\frac{1}{x-\alpha}$, corresponds to a vector having the same direction as the vector $x - \alpha$ but with a length equal to the reciprocal of $|x - \alpha|$. This is precisely the vector which represents the force at x due to a particle of unit mass at α which repels with a force whose magnitude is equal to its mass divided by the distance. If, then, we take the conjugate of both sides of the identity of the logarithmic derivative, we have the theorem of Gauss: *The roots of* f'(x) *which are not also roots of* f(x) *are the points of equilibrium in the field of force due to particles of unit mass at the roots of* f(x), *each of which exerts a repulsion equal to its mass divided by the distance.*

From this result it is but a step, though one not taken for many years, to the polygon theorem of Lucas, now sufficiently well known to have a place in Osgood's "Lehrbuch der Funktionentheorie," but discovered and rediscovered, proved and reproved in most of the languages of Europe—and all the proofs are substantially the same! This ignorance of the work of others characterizes even some of the most important contributions in this field. Lucas, for example, seems to have considered himself the discoverer of the theorem of Gauss, which really antedates his work by many years.

The polygon theorem, in its usual form, is a theorem of relative distribution which states that the roots of the derived function $f'(x)$ lie within or on the perimeter of the smallest convex polygon (or line segment) which includes within itself or on its boundary all the roots of $f(x)$. This statement implies that there is but one such polygon (or line segment), which reduces to a point if $f(x)$ has all its roots coincident. In case the polygon of Lucas does not reduce to a line or a point, the only roots of $f'(x)$ on its perimeter are multiple roots of $f(x)$. An equivalent form giving a separation theorem for the roots of $f(x)$ states that every straight line through a root of $f'(x)$ either passes through all the roots of

$f(x)$ or else separates them, *i. e.,* has roots on each side of it. This form is immediately suggested by the corresponding mechanical system; it is evident that a point of equilibrium must either be collinear with all the repelling particles, or else the latter must be seen under an angle of more than 180° from the former.

This result is only one of many concerning the relative distribution of roots of $f(x)$ and $f'(x)$ that may be inferred from the conditions of equilibrium of our mechanical system; we have deduced it by taking account only of the directions of the repelling forces. By considering their magnitudes as well J. Nagy (Jahresbericht der Deutschen Mathematiker Vereinigung, Vol. 27 (1918), page 44) has obtained a number of interesting theorems of which the following is one of the most striking: *If* α *is a root of the polynomial* f(x) *of degree* n, *and* β *is a root of* f'(x), *every circle through the points* β *and* $\gamma = \beta + (n-1)(\beta - \alpha)$ *contains at least one root of* f(x). The proofs given do not, however, make explicit use of the mechanical analogy. In a paper read before the International Congress of Mathematicians at Strasbourg J. L. Walsh has utilized Gauss's theorem in discussing the case where the roots of $f(x)$ lie in two circles.

If the repelling particles exert a force inversely proportional to the square of the distance we obtain theorems of relative distribution of roots in which $f'(x)$ is replaced by $f(x)f''(x) - [f'(x)]^2$; from a root of the latter function the roots of $f(x)$ must be seen under an angle of at least 90°, and the polygon of Lucas is replaced by one bounded by arcs of circles. Other extensions of this sort suggest themselves, but nothing, so far as I am aware, has been published along this line.

An immediate corollary of the polygon theorem states that all the roots of all the derived functions lie within the polygon of Lucas. It is well known that the centroid of the roots of $f(x)$ coincides with that of the roots of its derivative of any order. An often discovered theorem places the roots of $f'(x)$ at the foci of a curve determined by the roots of $f(x)$.

In 1912 Jensen, in a very suggestive memoir on the theory of equations (Acta Mathematica, Vol. 36), stated without proof a theorem for equations all of whose coefficients are real which may be regarded as an improvement on the polygon theorem. If $f(x)$ is a real polynomial its complex roots form conjugate pairs. The resultant force of repulsion due to particles at such a pair of points is directed away from the real axis at a point not on this axis and which lies outside the circle whose diameter is the line segment joining the pair; we designate this circle the *Jensen circle* of the pair. At a point within the Jensen circle and not on the real axis the resultant force due to the pair is directed toward the real axis, while on the real axis and on the circumference of the circle it is parallel to the real axis. Thus at a point which is neither on the axis of reals nor within or on the circumference of any of the Jensen circles corresponding to the complex roots, the resultant force of repulsion due to the whole system of particles at the roots of $f(x)$ cannot vanish, for the force due to each particle on the real axis is directed away from that axis, and the same is true of the forces due to pairs of particles at the complex roots. We thus have Jensen's theorem: *The roots of* f'(x) *which are not real must lie within or on the Jensen circles of* f(x). To be more precise, a root of $f'(x)$ cannot lie on a Jensen circle unless it is real, or unless it is a multiple root of $f(x)$, or unless it is also within or on another Jensen circle.

Since the addition of a constant force parallel to the real axis does not change the above argument, Jensen's theorem remains valid when we substitute for $f'(x)$ the function $af(x) + f'(x)$ where a is any real number. Another extension indicated by Jensen concerns the regions within which roots of the successive derived equations lie, these regions being defined in terms of the roots of $f(x)$. Thus the complex roots of $f''(x)$ are in the Jensen circles of $f'(x)$, whose centers are on the axis of reals and whose vertical diameters are within the Jensen circles of $f(x)$. The solution of a simple problem in envelopes shows that all the complex roots of $f''(x)$ lie within or on ellipses each of which has a pair of complex roots of $f(x)$ at the ends of its minor axis and has a major axis whose length is $\sqrt{2}$ times that of its minor axis. For the

rth derived equation the result is the same except that the ratio of lengths of axes is $\sqrt{r}$. Jensen states that this is also true of the function $g(D) . f(x)$, where $g(D)$ is a linear differential operator of order r with constant coefficients whose factors are all real, and that $f(x)$ may be an integral transcendental function of genus zero or one.

In a recent paper (Annals of Mathematics, Vol. 22 (1920) p. 128), J. L. Walsh notes some results for non-real polynomials which follow from considerations that led to Jensen's theorem. He also gives an answer to the question which at once suggests itself as to how many roots of $f'(x)$ lie within a Jensen circle when $f(x)$ is real by a method of interest in itself, doubtless suggested by Bôcher's treatment of a similar problem which we shall note later. By allowing all the roots of $f(x)$ outside a Jensen circle to move out to infinity, noting what roots of $f'(x)$ may enter or leave the circle, and counting those within the circle at the end of the process, Walsh concludes that *if a Jensen circle has on or within it* k *roots of* f(x) *and is not interior to nor has a point in common with any exterior Jensen circle, then it has on or within it not more than* $k+1$ *nor less than* $k-1$ *roots of* $f'(x)$. In a paper not yet published I have obtained a result a little more precise than this in which, for the sake of simpler statement, I will suppose neither $f(x)$ nor $f'(x)$ has multiple roots. By the term "root of even index" I designate a real root of $f'(x)$ between which and the next real root of $f(x)$ to the right or left there lies an odd number of real roots of $f'(x)$; if $f(x)$ has no real roots this term denotes every other real root of $f'(x)$, starting with the least. All the real roots of even index of $f'(x)$ can be shown to lie in or on Jensen circles, and every such circle that has no point in or on it within or on any other Jensen circle has within it either just one real root of even index of $f'(x)$, or just one pair of complex roots of $f'(x)$. The region covered by a system of Jensen circles each of which overlaps or touches some other of the system has within it the total number of real roots of even index and of pairs of complex roots of the derived equation which the circles would have if they were separated, but there may be circles of the system containing no such points. General criteria to determine whether even an isolated Jensen circle contains a pair of complex roots or a real root of even index of $f'(x)$ are lacking, though Walsh discusses special cases, in some of which we may use a circle smaller than Jensen's.

Relative distributions of the roots of a real polynomial $f(x)$ and of its derivative in various special cases have been discussed by H. B. Mitchell (Transactions of the American Mathematical Society, Vol. 19 (1918), p. 43). The identity of the logarithmic derivative is used, but the mechanical analogy and Jensen's theorem are not cited.

So far we have been concerned only with theorems of relative distribution for the roots of a polynomial and of its derivative. In a most suggestive paper by Bôcher ("A Problem in Statics and its Relation to Certain Algebraic Invariants," Proceedings of the American Academy of Arts and Sciences, Vol. 40 (1904), p. 469) our mechanical system is generalized by assigning to particles at points $e_1, e_2, \ldots e_n$ masses $m_1, m_2, \ldots m_n$ respectively, with the same law of repulsion as before. Negative values for the masses are admitted, the repulsion becoming an attraction in the case of the corresponding particles. The field of force is then given in both magnitude and direction by

$$K\left(\frac{m_1}{x-e_1}+\frac{m_2}{x-e_2}+\cdots\cdots+\frac{m_n}{x-e_n}\right).$$

The cases of greatest interest are those in which the sum of the masses is zero. By projecting such a system stereographically upon a sphere (the same result could be established by inversion on a circle about x), Bôcher proves that a point cannot be a position of equilibrium if it is possible to draw a circle through it upon which not all the particles lie and which completely separates the attractive particles which do not lie on it from the repulsive particles which do not lie on it.

A remarkable property of these systems whose total mass is zero is now developed by introducing homogeneous variables

$$x=\frac{x_1}{x_2},\ e_i=\frac{e'_i}{e''_i}.$$

If the above expression for the field of force is reduced to a common denominator within the parenthesis, the numerator is the product of x_2^2 and *a covariant* ϕ *of weight* 1 *of the* n *linear forms* $e_i''x_1 - e_i'x_2$. The points of equilibrium are roots of the covariant ϕ, and ϕ vanishes at no other points unless two of the particles coincide. If the points e_i are defined as the roots of a system of binary forms f_r, the masses of all the particles corresponding to each f_r being equal, ϕ is an integral rational covariant of the forms f_r, and we are thus led to theorems of relative distribution for the roots of a system of forms and those of a covariant of the system. In particular, if the system consists of but two forms, the covariant ϕ is their Jacobian; in all cases ϕ can be expressed as a polynomial in the ground-forms and Jacobians of pairs of the ground-forms.

The conditions of equilibrium of the corresponding mechanical system can now be interpreted as theorems of separation for the roots of the forms. Thus if f_1 and f_2 are two binary forms whose roots are all in circles C_1 and C_2 respectively, and these circles do not touch or overlap, then all the roots of the Jacobian of f_1 and f_2 are in C_1 and C_2. The actual number of roots in each circle is obtained by allowing the roots of f_1 to coalesce at a point a_1 and shrinking C_1 to this point; during this process C_1 is always to include all the roots of f_1. At the end of this process the Jacobian has $p_1 - 1$ roots at a_1, where p_1 is the degree of f_1. We conclude that the Jacobian originally had this number of roots in C_1, and a correspondingly determined number in C_2. The circles C_1 and C_2 may be replaced by circle-arc polygons.

The polygon theorem of Lucas corresponds to the special case where one of the ground-forms reduces to x_2.

A case of especial interest is that where one of the two ground-forms is linear; we have just noted a particular instance. The Jacobian of $y_2x_1 - y_1x_2$ and $f(x_1, x_2)$ is the first polar of (y_1, y_2) with respect to f. In a series of papers dating from 1874, to be found in his collected works, Laguerre had developed separation theorems for a binary form and its polars, without the use of our mechanical analogy. Bôcher seems to have been unacquainted with these results, which, however, are directly obtainable from his own. If the circle C_1 of the preceding paragraph is replaced by the point (y_1, y_2), we have Laguerre's theorem which states that if this point is outside a circle C_2 that contains all the roots of $f(x_1, x_2)$, then all the roots of the polar $y_1f'_{x_1} + y_2f'_{x_2}$ lie within C_2. Laguerre gives this a more striking form by supposing (x_1, x_2) taken arbitrarily and determining the "derived point" (y_1, y_2) as the point which makes the polar vanish. *Every circle through a point and its derived point either has all the roots of* $f(x_1, x_2)$ *on it, or else there is at least one root within and at least one root without the circle.* In non-homogeneous variables the derived point y of a point x with respect to $f(x)$ *is*

$$y = x - n\frac{f(x)}{f'(x)},$$

where n is the degree of $f(x)$. The first approximation to a root of $f(x)$ being x, the next approximation by Newton's method is $x - \frac{f(x)}{f'(x)}$. Thus we have a most interesting light upon Newton's method in the complex plane; it replaces x by a point *within* a circle on which x lies, and which surely contains a root of $f(x)$.

A point coincides with its derived point when and only when the point is a root of $f(x)$. Let α be such a simple root, and let β be its derived point with respect to $F(x)$, where $f(x) = (x - \alpha)F(x)$, and the degree of $f(x)$ is at least two. Since $F(\alpha) = f'(\alpha)$, and $F'(\alpha) = \frac{1}{2}f''(\alpha)$, we have

$$\beta = \alpha - (n-1)\frac{F(\alpha)}{F'(\alpha)} = \alpha - 2(n-1)\frac{f'(\alpha)}{f''(\alpha)}.$$

Each circle through α and β either has all the roots of $f(x)$ upon it or else at least one is within it and at least one is without. There is thus at least one root whose distance from α is not greater than $2(n-1)\left|\frac{f'(\alpha)}{f''(\alpha)}\right|$.

Laguerre and others have made interesting applications of these results to polynomials all of whose roots are real, and to polynomial solutions of linear differential equations.

Before leaving this phase of our subject we may note, with Laguerre, that similar theorems hold for each of the successive polars of a binary form with respect to a point. An interesting field hardly touched as yet is that of separation theorems for the successive polars of a form with respect to a sequence of points defined as the roots of another form. By taking the two forms in a special case where they are apolar Grace has proved (Proceedings of the Cambridge Philosophical Society, Vol. 11 (1901), p. 35) a result equivalent to this: *If the distance apart of two roots* α_1, α_2 *of a polynomial* $f(x)$ *of degree* n *is* 2a, *there is at least one root of* $f'(x)$ *on or in the circle whose radius is* $a \cot \frac{\pi}{n}$, *and whose center is* $\frac{1}{2}(a_1 + a_2)$. In this paper lack of references indicates ignorance of Laguerre's work. The same result was proved later by Heawood (*Quarterly Journal of Mathematics,* Vol. 38 (1907), p. 84) by allowing all the other roots of $f(x)$ to vary suitably. Here, again, there is no reference to any other work in this field.

To return to more recent work on the vanishing of the Jacobian of two forms f_1 and f_2, we note two very interesting papers by Walsh in the Transactions of the American Mathematical Society, in which are discussed cases where the roots of the ground-forms are in three circles, instead of two. An added interest is shown to attach to the Jacobian because the numerator of the derivative of a rational function

$$\frac{u(x)}{v(x)} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

is x_2^2 multiplied by the Jacobian of f_1 and f_2. Separation theorems for the Jacobian are then interpretable in terms of this derivative. The results of these papers are, of course, only a first step to the consideration of still more general separation theorems. The field is the more interesting in that its investigation involves a combination of mechanical, algebraical, and geometrical considerations.

I must close with only a mention of certain extensions of the problem we have so far considered. Thus Bôcher, generalizing a method due to Stieltjes, considers the positions of equilibrium of a system of free particles of equal mass in a field of force due not only to a number of fixed repelling particles, but also to their own mutual repulsions according to the same law. If the total mass of fixed and moving particles is 1, the positions of equilibrium of the free particles are determined by the vanishing of covariants, of which some examples are given by Bôcher. These results, as well as some obtained by adding a force function $K[f(x)]$, are useful in the study of polynomial solutions of differential equations. We must regret that Bôcher was never able to fulfill the hope twice expressed in this paper that he might be able to return in detail to these problems which he had merely sketched. Their investigation requires considerable skill, but, if successful, would add a new and important chapter to algebra, with a striking application of invariant theory.

D. R. CURTISS
NORTHWESTERN UNIVERSITY

WILLIAM BATESON ON DARWINISM

ASIDE from the fine impression created by the admirable series of papers and addresses in biology, zoology and genetics in Toronto at the Naturalists' meeting, a very regrettable impression was made by a number of passages in the addresses of Professor William Bateson, the distinguished representative of Cambridge University and British biology. On the morning following his principal address the Toronto *Globe* (December 29, 1921) published, in large letters: "Bateson Holds That Former Beliefs Must Be Abandoned. Theory of Darwin Still Remains Unproved and Missing Link Between Monkey and Man Has Not Yet Been Discovered by Science. Claims Science Has Outgrown Theory of Origin of Species." In intermediate type it announced: "Distinguished Biologist from Britain Delivers Outstanding Address on Failure of Science to Support Theory That Man Arrived on Earth Through Process of Natural Selection and Evolution of Species. Have Traced Man Far Back but Still He Remains Man," and, in smaller type: **The missing link is still missing, and the Dar-**