SCIENCE

FRIDAY, JANUARY 16, 1920

CONTENTS

MSS. intended for publication and books, etc., intended for review should be sent to The Editor of Science, Garrison-on-Hudson, N. Y.

RECENT ADVANCES IN DYNAMICS[1]

A HIGHLY important chapter in theoretical dynamics began to unfold with the appearance in 1878 of G. W. Hill's researches in the lunar theory.

To understand the new direction taken since that date it is necessary to recall the main previous developments. In doing this, and throughout, we shall refer freely for illustration to the problem of three bodies.

The concept of a dynamical system did not exist prior to Newton's time. By use of his law of gravitation Newton was able to deal with the Earth, Sun, and Moon as essentially three mutually attracting particles, and by the aid of his fluxional calculus he was in a position to formulate their law of motion by means of differential equations. Here the independent variable is the time and the dependent variables are the nine coordinates of the three bodies. Such a set of ordinary differential equations form the characteristic mathematical embodiment of a dynamical system, and can be constructed without especial difficulty.

The aim of Newton and his successors was to find explicit expressions for the coordinates in terms of the time for various dynamical systems, just as Newton was able to do in the problem of two bodies. Despite notable successes, the differential equations of the problem of three bodies and of other analogous problems continued to defy "integration."

Notwithstanding the lack of explicit expressions for the coordinates, Newton was able to treat the lunar theory from a geometrical point of view. Euler, Laplace, and others invented more precise analytical methods based upon series. In both cases the bodies which are disturbing the motion of the

[1] Address of the vice-president and chairman of Section A—Mathematics and Astronomy—American Association for the Advancement of Science, St. Louis, December, 1919.

Moon are assumed first to move in certain periodic orbits, and the perturbations of the Moon are assumed to be the same as if the other bodies did move in such hypothetical orbits. The principle of successive approximations characterizes these methods.

The chief other advance made was based on the following principle: if a function is a maximum or minimum when expressed in terms of one set of variables it is also a maximum or minimum for any other set; hence, if the differential equations of dynamics can be looked upon as the equations for a maximum or minimum problem, this property will persist whatever variables be employed. This principle, developed mainly by Lagrange, W. R. Hamilton, and Jacobi, enables one to make the successive changes of variables required in the method of successive approximations by merely doing so in a single function.

Here too the results are chiefly of formal and computational importance.

The last great figure of this period is Jacobi. His "Vorlesungen über Dynamik" published in 1866 represents a highwater mark of achievement in this direction.

Nearly all fields of mathematics progress from a purely formal preliminary phase to a second phase in which rigorous and qualitative methods dominate. From this more advanced point of view, inaugurated in the domain of functions of a complex variable by Riemann, we may formulate the aim of dynamics as follows: to characterize completely the totality of motions of dynamical systems by their qualitative properties.

In Poincaré's celebrated paper on the problem of three bodies, published in 1889, where he develops much that is latent in Hill's work, Poincaré proceeds to a treatment of the subject from essentially this qualitative point of view.

A first notion demanding reconsideration was that of integrability, which had played so great a part in earlier work. In 1887 Bruns had proved that there were no further algebraic integrals in the problem of three bodies. Poincaré showed that in the so-called restricted problem there were no further integrals existing for all values of a certain parameter and in the vicinity of a particular periodic orbit. Later (1906) Levi-Civita has pointed out that there are further integrals of a similar type in the vicinity of part of any orbit.

Thus it has become clear that the question as to whether a given dynamical problem is integrable or not depends on the kind of definition adopted. However, the most natural definitions have reference to the vicinity of a particular periodic motion. The introduction of a parameter by Poincaré is to be regarded as irrelevant to the essence of the matter.

From the standpoint of pure mathematics, a just estimate of the results found in integrable problems may be obtained by reference to the problem of two bodies, or, more simply still, of the spherical pendulum. The integration by means of elliptic functions shows that the pendulum bob rotates about the vertical axis of the sphere through a certain angle in swinging between successive highest and lowest points. But the form of the differential equation renders this principal qualitative result self-evident, while the most elementary existence theorems for differential equations assure one of the possibility of explicit computation. Hence the essential importance of carrying out the explicit integration lies in its advantages for purposes of computation.

The series used in the calculations of the lunar theory and other similar theories were given their proper setting by Poincaré. He showed that they were in general divergent, but were suitable for calculation because they represented the dynamical coordinates in an asymptotic sense.

The fact that the first order perturbations of the axes in the lunar theory can be formally represented by such trigonometric series had led astronomers to believe that the perturbations remained small for all time. But the fact of divergence made the argument for stability inconclusive.

It is easy to see that this question of stability, largely unsolved even to-day, is of fundamental importance from the point of view formulated above. For, in a broad sense, the question is that of determining the general character of the limitations upon the possible variations of the coordinates in dynamical problems.

We wish to mention briefly four important steps in advance in this direction.

The first is due to Hill who showed in his paper that, in the restricted problem of three bodies, with constants so chosen as to give the best approximation for the lunar theory, the Moon remains within a certain region about the Earth, not extending to the Sun. In fact here there is an integral yielding the squared relative velocity as a function of position, and the velocity is imaginary outside of this region.

In his turn, Poincaré showed that stability exists in another sense, namely for arbitrary values of the coordinates and velocities there exist nearby possible orbits of the Moon which take on infinitely often approximately the same set of values. His reasoning is extremely simple, and is founded on a hydrodynamic interpretation in which the orbits appear as the stream lines of a three-dimensional incompressible fluid of finite volume in steady motion. A moving molecule of such a fluid must indefinitely often partially reoccupy its original position with indefinite lapse of time, and this fact yields the stated conclusion.

In 1901 under the same conditions Levi-Civita proved that, if the mean motions of the Sun and Moon about the Earth are commensurable, instability exists in the following sense: orbits as near as desired to the fundamental periodic lunar orbit will vary from that periodic orbit by an assignable amount after sufficient lapse of time. This result, which is to be anticipated from the physical point of view, makes it highly probable that instability exists in the incommensurable case also.

These three results refer to the restricted problem of three bodies.

Finally there is Sundman's remarkable work on the unrestricted problem contained in his papers of 1912 and of earlier date. Lagrange had proved that if a certain energy constant is negative, the sum of the mutual distances of the three bodies becomes infinite. Sundman showed that, even if this constant is positive, the sum of the three mutual distances always exceeds a definite positive quantity, at least if the motion is not essentially in a single plane. Thus he incidentally verified a conjecture of Weierstrass that the three bodies can never collide simultaneously. These and other results seem to me to render it probable that *in general* the sum of the three distances increases indefinitely. Thus, if this conjecture holds, in that approximation where the Earth, Sun and Moon are taken as three particles, the Earth and Moon remain near each other but recede from the Sun indefinitely. The situation is worthy of the attention of those interested in astronomy and in atomic physics.

As we have formulated the concept of stability, it is essentially that of a permanent inequality restricting the coordinates. We may call a dynamical system transitive in a domain under consideration if motions can be found arbitrarily near any one state of motion of the domain at a particular time which pass later arbitrarily near any other given state. In such a domain there is instability. If we employ the hydrodynamic interpretation used above, the molecule of fluid will diffuse throughout the corresponding volume in the transitive case, and will diffuse only partially or not at all in the intransitive case. The geodesics on surfaces of negative curvature, treated by Hadamard in 1898, furnish a simple illustration of a transitive system, while the integrable problem of two bodies yields an intransitive system. Probably only under very special conditions does intransitivity arise.

It is an outstanding problem of dynamics to determine the character of the domains within which a given dynamical system is transitive.

A less difficult subject than that of stability is presented by the singularities of the motions such as arise in the problem of three bodies at collision. The work of Levi-Civita and Sundman especially has shown that the singularities can frequently be eliminated by means of appropriate changes of variables. In consequence the coordinates of dynamical systems admit of simple analytic representation for all values of the time. In particular Sundman has proved that the coordinates and the time in the problem of three bodies can be expressed in terms of permanently convergent power series, and thus he has "solved" the problem of three bodies in the highly artificial sense proposed by Painlevé in 1897. Unfortunately these series are valueless either as a means of obtaining qualitative information or as a basis for numerical computation, and thus are not of particular importance.

From early times the mind of man has persistently endeavored to characterize the properties of the motions of the stars by means of periodicities. It seems doubtful whether any other mode of satisfactory description is possible. The intuitive basis for this is easily stated: any motion of a dynamical system must tend with lapse of time towards a characteristic cyclic mode of behavior.

Thus, in characterizing the motions of a dynamical system, those of periodic type are of central importance and simplicity. Much recent work has dealt with the existence of periodic motions, mainly for dynamical systems with two degrees of freedom.

An early method of attack was that of analytical continuation, due to Hill and Poincaré. A periodic motion maintains its identity under continuous variation of a parameter in the dynamical problem, and may be followed through the resultant changes. G. Darwin, F. R. Moulton and others have applied this method to the restricted problem of three bodies. Symmetrical motions can be treated frequently by particularly simple methods. Hill made use of this fact in his work.

Another method is based on the geodesic interpretation of dynamical problems. This has been developed by Hadamard, Poincaré, Whittaker, myself, and others. The closed geodesics correspond to the periodic motions, and the fact that certain closed geodesics of minimum length must exist forms the basis of the argument in many cases. As an example of another type, take any surface with the connectivity of a sphere and imagine to lie in it a string of the minimum length which can be slipped over the surface. Clearly in being slipped over the surface there will be an intermediate position in which the string will be taut and will coincide with a closed geodesic.

Finally there is a less immediate method of attack which Poincaré introduced in 1912, and which I have tried to extend. By it the existence of periodic motions is made to depend on the existence of invariant points of certain continua under one-to-one continuous transformation. The successful application of this method involves a preliminary knowledge of certain of the simpler periodic motions.

Periodic motions fall into two classes which we may call hyperbolic and elliptic. In the hyperbolic case analytic families of nearby motions asymptotic to the given periodic motion in either sense exist, while all other nearby motions approach and then recede from it with the passing of time. In the elliptic case the motion is formally stable, but the phenomenon of asymptotic families not of analytic type arises unless the motion is stable in the sense of Levi-Civita.

In a very deep sense the periodic motions bear the same kind of relation to the totality of motions that repeating doubly infinite sequences of integers 1 to 9 such as

. . . 2323 . . .

do to the totality of such sequences.

In trying to deal with the totality of possible types motion it seems desirable to generalize the concept of periodic motion to recurrent motion as follows: any motion is recurrent if, during any interval of time in the past or future of sufficiently long duration T, it comes arbitrarily near to all of its states of motion. With this definition I have proved that every motion is either recurrent or approaches with uniform frequency arbitrarily near a set of recurrent motions.

The recurrent motions correspond to those double sequences specified above in which every finite sequence which is present at all occurs at least once in every set of N successive integers of the sequence.

In any domain of transitivity the two extreme types of motion are the recurrent motions on the one hand and the motions which pass arbitrarily near every state of motion in the domain on the other. Both types necessarily exist, as well as other intermediate types.

The precise nature of such recurrent motions has yet to be determined, but Dr. H. C. M. Morse in his 1918 dissertation at Harvard has shown that there exists non-periodic recurrent motions of entirely new type in simple dynamical problems.

Such are a few of the steps in advance that theoretical dynamics has taken in recent years. I wish in conclusion to illustrate by a very simple example the type of powerful and general geometric method of attack first used by Poincaré.

Consider a particle P of given mass in rectilinear motion through a medium and in a field of force such that the force acting upon P is a function of its displacement and velocity. In order to achieve simplicity I will assume further that the law of force is of such a nature that, whatever be the initial conditions, the particle P will pass through a fixed point O infinitely often.

If P passes O with velocity v it passes O at a first later time with a velocity v_1 of opposite sign. We have then a continuous one-to-one functional relation $v_1 = f(v)$. If v is taken as a one-dimensional coordinate in a line, then the effect of the transformation $v_1 = f(v)$ is a species of qualitative "reflection" of the line about the point O.

If this "reflection" is repeated the resultant operation gives the velocity of P at the second passage of O, and so on. But the most elementary considerations show that either (1) the reflection thus repeated brings each point to its initial position, or (2) the line is broken up into an infinite set of pairs of intervals, one on each side of O, which are reflected into themselves, or (3) there is a finite set of such pairs of intervals, or (4) every point tends toward O (or away from it) under the double reflection.

Hence there are four corresponding types of systems that may arise. Either (1) every motion is periodic and O is a position of equilibrium, or (2) there is an infinite discrete set of periodic motions of increasing velocity and amplitude (counting the equilibrium position at O as the first) such that, in any other motion, P tends toward one of these periodic motions as time increases and toward an adjacent periodic motion in past time, or (3) there is a finite set of periodic motions of similar type such that, in any other motion, P behaves as just stated, if there be added a last periodic motion with "infinite velocity and amplitude" as a matter of convention, or (4) in every motion P oscillates with diminishing velocity and amplitude about O as time changes in one sense and with ever increasing velocity and amplitude as time changes in the opposite sense.

Here we have used the obvious fact that there is a one-to-one correspondence between velocity at O and maximum amplitude in the immediately following quarter swing.

This example illustrates the central rôle of periodic motions in dynamical problems. It is also easy to see in this particular example that the totality of motions has been completely characterized by these qualitative properties in a certain sense which we shall not attempt to elaborate.

What is the place of the developments reviewed above in theoretical dynamics?

The recent advances supplement in an important way the more physical, formal, and computational aspects of the science by providing a rigorous and qualitative background.

To deny a position of great importance to these results, because of a lack of emphasis upon the older aspects of the science would be as illogical as to deny the importance of the concept of the continuous number system merely because of the fact that in computation attention is confined to rational numbers.

GEORGE D. BIRKHOFF

SIR WILLIAM OSLER (1849–1919)

AFTER a tedious and painful illness, Sir William Osler, Regius professor of medicine at Oxford, died at his home in Norham Gardens on December 9, 1919. In spite of in-