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THE RELATION OF MATHEMATICS TO THE NATURAL SCIENCES[1]

IN considering the relationship of mathematics to the natural sciences, we shall do well to see what mathematics is and what are its methods.

Mathematics has not always been looked at through the same glasses. The field of mathematics to the early workers was number and quantity. Euclid put into his axioms what he considered to be the fundamental facts of the world about him. Diophantus, of Alexandria, a worker in algebra, considered only positive roots of equations. They were dealing with realities and not with abstract matters. Some time later mathematicians tried to prove their axioms—often called self-evident truths—and made a wonderful discovery. That was, that a "self-evident truth" might be replaced by its contrary and the result still be a consistent body of doctrine. And thus the glasses were changed, to be mathematical the conclusions must be the result of the assumptions and these must be consistent. The assumptions need have no physical interpretation, indeed they might contradict any of our theories, but they must not contradict each other. There might be foreign war, but no internal conflict. I like the following of Professor Keyser, of Columbia University:[2]

> He (the mathematician) is not absolutely certain, but he believes profoundly that it is possible to find axioms, sets of a few propositions each, such that the propositions of each set are compatible, that the propositions of such a set imply other propositions, and that the latter can

[1] Read before the Purdue University chapter of Sigma Xi, October 25, 1916.

[2] SCIENCE, Vol. 35 (1912), p. 107.

be deduced from the former with certainty. That is to say, he believes that there are systems of coherent or consistent propositions, and he regards it as his business to discover such systems. Any such system is a branch of mathematics.

A word might be said about pure and applied mathematics. We may have a branch of mathematics with its postulates or axioms consistent and have no physical interpretation of them. On the other hand, we may make our postulates consistent with what we believe to be the proper interpretation of certain phenomena, and this would be applied mathematics. It is to be observed that after we have our conditions once fixed by our interpretation of these phenomena, we proceed to our conclusions in a way which is wholly independent of whether we have the right interpretation or not, and are thus back in the domain of pure mathematics.

The popular conception of mathematics has been that it devoted itself to problem solving. You will see, however, that the mathematician concerns himself not with the solution of particular problems, but with the principles which underlie the solution of classes of problems. There is and has been a lively interest in problem solving as is evidenced by the problem departments of various journals. To some the solution of these problems has offered simply the diversion which comes from the solution of a puzzle, to others they have offered a real mathematical stimulus.

There are two general methods of working—I mean of research—in mathematics, the intuitional and postulational. In the case of the first the worker jumps to his conclusions, as it were, guided by some analogy or by his sense of what the facts should be or by his experience; and then follows this drawing of conclusions by filling in his proof by rigorous deduction. In the second method the postulates are kept definitely in view and results are reached by deduction. Most discoveries are, I think, made by the intuitional method. Most progress can be made by leaping across barriers and viewing the country beyond and then returning to build roads and tunnels. It is true that when we attempt to build the road it may not lead us where we leaped, it may not lead us anywhere, and we must return to our starting point. But we build with so much more enthusiasm, with so much more skill, if we think we know where the road leads. The postulational method of work is more formal and is a better tool for the road building than for spying out the land.

We learned our arithmetic by the intuitional method. We said $1 + 1 = 2$, not because of some set of postulates, but because in our experience one and one gave something to which we attached the name two. Now to set down our postulates and prove that $1 + 1 = 2$ is possible and profitable at the proper time, but altogether out of place in an elementary arithmetic. In plane geometry we had our introduction to the postulational method. In this subject we started with a set of postulates explicitly stated and deduced from them certain results. In discovering the facts of Euclidean geometry, intuition is largely called upon, while in setting those facts down in a text-book we use the postulational method. Euclidean geometry is so largely intuitional in discovery because its postulates agree with our notions of space. In the non-Euclidean geometries we can not trust our intuition and must depend directly on our postulates.

If instead of saying that the whole is equal to the sum of its parts, we say that a part may equal the whole, our intuition is no safe guide. Other examples might be given.

Research work in mathematics attracts two classes of workers, those interested for mathematics' sake and those interested in creating a tool with which to attack some other science. The search after truth—geographical, religious, scientific—has always lured men. The desire to create, to build some new thing, is continually finding outlet in invention, in exploration, and in scientific research. That desire which sends some men to the poles of the earth, to the tops of mountains, or to the heart of the desert, sends other men over the mountain tops of geometry, or among the pitfalls of analysis, or through the labyrinth of point sets, to some hitherto untrodden field of mathematics. The mathematician creates an intellectual fabric which is just as real and just as beautiful to him as the tapestry is to the weaver. Some put forth their effort in any field that attracts; others, the utilitarians, choose parts which they think will be fruitful in applications.

Knowledge of pure science precedes its application. The properties of conic sections were well known before Kepler and Newton wanted to use them in their theories of planetary motion. The infinitesimal calculus was developed before and not after it was needed in physics. The differential equation had to be understood before it could be applied. Mathematicians have ready now the integral equation and the difference equation which, I believe, have only made a beginning in their service to science. It may be the man who is not seeking utilitarian ends who discovers the most useful facts. Roentgen was not seeking an aid for the medical profession when he discovered the X-rays. That man who reads carefully the history of scientific discovery and its application will not criticize any worker for choosing a field which is apparently remote from usefulness.

How many are working in mathematics, what have they done and what are they doing? There are some six or eight of the more important mathematical societies in various parts of the world with a total membership of over three thousand. These societies comprise in their membership practically all the research workers, besides many others not so engaged. The Subject Index of the Royal Society of London Catalogue of Scientific Papers, volume 1, which gives practically a complete list of mathematical articles which appeared during the nineteenth century, says in its preface that it contains 38,748 entries referring to articles in 701 serials and has rejected 750 as having no scientific value. G. Valentin, of Berlin, has collected a list of 150,000 titles of books and articles published before the beginning of the twentieth century. The *Jahrbuch über die Fortschritte der Mathematik* is a yearly review and each year publishes a volume of about 1,000 pages with very short reviews of books and of papers published in about 200 serials. A very conservative estimate would be that each year there appear 2,000 articles, in addition to the books which contain no new matter. Professor G. A. Miller, of Illinois University, estimates that there was published during the first fifteen years of the present century about one fifth as much mathematical research as during all time before. Mathematicians have varied greatly in their productivity. At one extremity is Galois, killed in a duel before he was twenty-one years old, whose essential contribution to mathematics requires about sixty pages of print; and at the other are Cauchy, whose works are expected to fill twenty-seven volumes when printed, and Euler, the printing of whose work as planned will fill forty-five large volumes.

Now, what relation can this science which deals with the abstract have to do with the natural sciences which deal with the concrete? Professor A. Voss, of the University of Munich, said in a lecture in 1908:

Our entire present civilization, as far as it depends upon the intellectual penetration and utilization of nature, has its real foundation in the mathematical sciences.

You will observe that he does not say in mathematics, but in the sciences which have made use of mathematics in their development. Let us investigate this a little. Can you realize what would happen, just what stage of civilization we should be in, if all that is developed by the use of mathematics could be removed from the world by some magic gesture? Every branch of physics makes use of mathematics; chemistry is not free from it; engineering is based upon its developments; sociology, economics and variation in biology make use of statistics and probability. Our skyscrapers must disappear; our great bridges and tunnels must be removed; our transportation systems, our banking systems, our whole civilization, indeed, must step backward many centuries.

Mathematics and its symbolism appear in rather unexpected places. S. G. Barton,[3] of the Flower Observatory, University of Pennsylvania, says that in the Encyclopædia Britannica, written not for the specialist so much as for the general reader, there are one hundred four articles which make use of the notation of the infinitesimal calculus, of which only about one fourth are pure mathematics. You may be surprised to know that you need the infinitesimal calculus to read the articles on clock, heat, lubrication, map, power transmission, ship building, sky, steam engine and strength of materials.

Take these sentences from Simon Newcomb's article in the Encyclopædia Britannica on celestial mechanics:

[3] SCIENCE, Vol. 40 (1914), p. 697.

The purpose of the present article is to convey a general idea of the methods by which the results of celestial mechanics are reached, without entering into those technical details which can be followed only by a trained mathematician. It must be admitted that any intelligent comprehension of the subject requires at least a grasp of the fundamental conceptions of analytical geometry and the infinitesimal calculus, such as only one with some training in these subjects can be expected to have. . . . The non-mathematical reader may possibly be able to gain some general idea, though vague, of the significance of the subject.

Sir John Herschel in his introduction to his book, "Outlines of Astronomy," says:

Admission to its (astronomy's) sanctuary and to the privileges and feelings of a votary is only to be gained by one means—sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range.

Professor and Mrs. Mittag-Leffler have given their fortune to the founding of an institute for the promotion of research in pure mathematics in Sweden and the other Scandinavian countries. They say:[4]

Our testament owes its origin to the lively conviction that a people that does not accord to mathematics a high place in its estimation, will never be in a position to fulfil the most lofty tasks of civilization, and to enjoy in consequence that international consideration which is itself, in the end, an effective means of preserving our place in the world and of safeguarding our right to live our own life.

I am not claiming any superiority for mathematics over the other sciences. I am trying to emphasize how indispensable mathematics has been in the development of other sciences. Wherein lies its worth?

Mathematics is an exact science, that is, with the conditions—the postulates—definitely given, the conclusion admits of no doubt, of no variation. The worker in the

[4] *Bulletin of the American Mathematical Society*, Vol. XXIII., No. 1 (1916), p. 31.

fields of the natural sciences sees the result —the conclusion—before him and tries to work back to underlying causes. Nature has laid a foundation and reared thereon a mighty superstructure, through which the natural scientist wanders amid a maze of halls and chambers, scratching the surface a little here and a little there trying to find what sort of a foundation can support all this that he sees. The natural scientist accepts as his foundation that theory which best explains the results. The theory may be wrong, but it serves all the purposes of a scientific theory if it explains to a fair degree of satisfaction observed phenomena. I seem to remember to have read the statement of a physicist that we should probably explain some phenomena of light on the wave theory and other phenomena on the atomic theory. Whenever a theory is contradicted by experiments, the natural scientist seeks another. This may seem a rather "hit or miss" way of scientific research, but it is the best that man can hope for with his human intellect trying to find first causes underlying the workings of a universe.

The mathematician is not thus restricted. He lays his own foundation. Some natural science may furnish the material for this foundation, but the mathematical mason handles each stone and sets it in proper relationship in the mortar of consistency.

By being an exact science, mathematics serves the natural sciences in two ways. In the first place, the methods of mathematical deduction offer a convenient means of testing the consistency of a theory. Mathematics will take the essential elements of a theory as postulates and deduce the necessary conclusions. If this leads to a contradiction of experiment, the incorrectness of the theory is shown. It might even be possible in certain cases to locate exactly what part of the theory is at fault. If the deduced results agree with experiment our faith in our theory is strengthened. An example of this sort of thing is to be found in Carmichael's "Theory of Relativity." The author, a mathematician, has taken as his postulates statements whose truth is accepted by a number of physicists. He has arrived, by purely mathematical means, at results whose truth or falsity are susceptible of experimental proof. The results of such an experiment as he suggests would disprove or increase our faith in the truth of his postulates.

A second way in which this exact science can serve the natural sciences, and which does not differ much from the way already mentioned, is in the matter of discovery. If the postulates of a natural theory are true, then all its consequences are true. Mathematics offers a tool for finding out these consequences. A classic example of discovery in this manner is Maxwell's prediction of the pressure due to heat or light radiation, which was not experimentally demonstrated for several years after Maxwell's death. Sir W. R. Hamilton's prediction of conical refraction is another such example. This prediction was experimentally verified by his colleague Lloyd within a short time after it was announced. This mathematical working out of the consequences of a theory has, in my judgment, not received its due at the hands of the natural scientist.

A more universal service rendered by mathematics has been the furnishing of a system of shorthand that is as exact and much more workable than the completely written out statement. If you do not believe in the value of a well-chosen symbolism, try to calculate the value of 22⅓ dozen eggs at 39½ cents per dozen by using Roman notation. A well-known example of the value of symbolism is furnished by mathematics itself in the development of the calculus. Newton and Leibnitz made independent discoveries in this field. Newton chose a rather clumsy notation, Leibnitz our present notation. The English mathematicians used the Newtonian notation and were hampered to such an extent that they fell far behind the continental mathematicians in the development of the calculus. The graph is an example of an almost universal scientific symbol for representing tabulated data. Some one has said that we capitalize our knowledge in an equation. The natural scientist finds a well-developed symbolism in mathematics and proceeds to make use of it without taking the trouble of creating one of his own.

I have spoken of the service of mathematics to the natural sciences. There is a return service. The natural sciences furnish a rich field for mathematical research. One of the problems that has called forth the efforts of many mathematicians in the recent past has been the three body problem. There are many others of lesser note and many more still untouched. In a number of the *Bulletin of the American Mathematical Society* for 1914, there appeared a letter from a Mr. Paaswell, an engineer, enumerating a number of engineering problems which he thinks the mathematician should attack. Physics and mathematics have acted and reacted upon each other to the enrichment of both. Witness the work of J. C. Maxwell.

The natural sciences and developments based upon them not only furnish a rich field for mathematical research, but a field which promises to quickly make mathematics of service to the world. The scientist in any field feels justified in his labors if he discovers new facts, whether or not they are immediately applicable to the problems of daily life. He hopes they will be serviceable some day. The investigator in pure mathematics may work for the pleasure he gets from his mental creations, but in most men there is the deeper purpose of serving the world. The natural sciences furnish a field for the choice of postulates, the development of the consequences of which gives prospect of practical worth in the immediate future.

The mathematician is always confronted by the question of the consistency of his postulates. If these are chosen from some natural science, he can often find some physical system in which his postulates are verified. This exhibition of an example in which the postulates are verified and which from the fact of its physical existence offers no contradictory conclusions, is the best proof of consistency.

In our colleges and technical schools the time the engineering student or student of pure science gives to mathematics is reduced to a minimum in order to make room for more technical subjects, and in our graduate schools the mathematical student is given little opportunity to study anything but pure mathematics. The result is that one group of students knows too little mathematics to develop properly their field of study and the other group knows too little of the natural sciences and their application to apply to them their knowledge of mathematics.

How can we remedy this? I do not know. Not every engineering student or student of pure science should be required to become proficient in mathematics, nor every mathematician be required to become an engineer. This would be a great waste of time and effort without commensurate returns. The sooner, however, some plan is worked out whereby the pure science or engineering student of marked mathematical ability is given a chance to develop that ability, or the mathematical student with a tendency to applied mathematics is given opportunity in that direction, the sooner will come the time of fullness of the development of applied science.

Mathematics has been a well-nigh indispensable tool in the development of the natural sciences and their applications. On the other hand the natural sciences and particular problems set by science have challenged the ability of mathematicians and spurred them on to the achievement of larger results in pure mathematics. Whoever can strike this flint of mathematics upon the steel of natural science and produce fire is doing the world service. The oftener fire is produced the greater will be the development of both mathematics and natural science.

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EDUCATION AFTER THE WAR

The sharp debate on the place of science in education which took place recently in the House of Lords between Lord Haldane on the one side and Lord Cromer and Viscount Bryce on the other side is an example of the kind of misunderstanding which it is necessary to eliminate if we here in the United States and you in England are to act wisely in the matter of education after the war.

In his sesquicentennial address at Princeton University nineteen years ago Woodrow Wilson said that if he was not mistaken the "scientific spirit" of the age is doing us a great disservice, working in us a certain great degeneracy; and yet he said that he had no indictment against science itself, but only a warning to utter against the atmosphere which has stolen from our laboratories and lecture rooms and into the general air of the world at large. It is a noxious intoxicating gas which has somehow got into the lungs of the rest of us, a gas which it would seem forms only in the outer air.

Now it is not easy even for one of Dr. Wilson's training to express himself with perfect clearness in a matter of this kind; and although we are in full sympathy with what we understand Dr. Wilson's point of view to be, we do not like his use of the term "scientific spirit." The true scientific spirit, the spirit of such men as Kelvin and Helmholtz, is beyond criticism; but the great things such men have done have brought upon us the most distressing and stupid form of idolatry the world has ever seen, and the men who have the true scientific spirit are the only men, as a rule, who are free from it.

Science is *finding out* and *learning how,* whereas most people think of science only in terms of its material results. These results have indeed fascinated the crowd, and the great majority of men have adopted a scale of physical values for everything in life "with a consequent neglect of quality and a denial of human value in everything. We have a philosophy of rectangular beatitudes and spherical benevolences, a theology of universal indulgence, a jurisprudence which will hang no rogues; all of which means, in the root, incapacity of discerning worth and unworth in anything and least of all in man. Whereas, nature and heaven command us, at our peril, to discern worth from unworth in everything and most of all in man."

"Our real problem now, as always, is 'Who is best man?' and the fates forgive much—forgive the wildest, fiercest and crudest experiments—if fairly made in the settling of that question. Theft and blood-guiltiness are not pleasing in their sight, and yet the favoring powers of the spiritual and material worlds will confirm to you your stolen goods, and their noblest voices applaud the lifting of your spear and rehearse the sculpture of your shield, if only your robbing and slaying have been done in fair arbitrament of that question 'Who is best man?' But if you refuse such inquiry you will come at last to face the same question wrong side upwards, and your robbing and slaying must then be done to find out 'Who is *worst* man?' which in so wide an order of inverted merit is indeed not easy."

This impassioned statement of a great English writer and moralist seems to us to touch the essence of all unfriendliness towards the sciences among seriously thoughtful men, and although this unfriendliness is largely mis-