SCIENCE

FRIDAY, JULY 23, 1915

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MSS. intended for publication and books, etc., intended for review should be sent to Professor J. McKeen Cattell, Garrison-on-Hudson, N. Y.

FORTY YEARS' FLUCTUATIONS IN MATHEMATICAL RESEARCH[1]

IN the year 1870 was published the first volume of the *Jahrbuch über die Fortschritte der Mathematik,* a new venture in the mathematical world. It was intended to relieve students and investigators from the necessity of searching through all available recent books and serials in the quest of any particular topic, or in the effort to keep posted on the discoveries of current interest. Similar attempts have been made, both before and since 1870, in other fields of science; and their effect and influence have been so marked that even the general reader has now Poole's Index and its successors, attempting in a less minute way the same service for general periodical literature. Under each title of book or article the *Jahrbuch* gives a brief analysis, sometimes a criticism, so that the reader follows up only those articles whose actual content proves important for him.

At the outset the expectation was confidently expressed that the *Jahrbuch* would appear within six months after the close of its year. But serials were slow in reaching the office, there were from twenty to forty referees, and once a printer's strike interposed for half a year; so that three years often elapsed instead of six months. Twice a double volume was issued in the hope of lessening the delay; but it soon fell back, and now our most recent volume covers the year 1909. These voluminous handbooks, now covering forty-two years and occupying eight feet of shelf-room, offer op-

[1] Read before the Vassar Faculty Club, and the Columbia Mathematical Colloquium, February, 1914.

portunity for certain inquiries which without them would be futile; and first, for questions quite impersonal and general.

One's first impression, on inspecting this series, is that the number of authors of books and contributors to *Acta eruditorum, Berichte der Akademie, Comptes rendus, Transactions* or *Journals* must have increased surprisingly during forty years. So we count sample volumes, say those for 1875 and 1905. In the former, the number of serials reviewed and the number of writers were, respectively, about 110 and 510; in the latter volume these numbers are 180 and 1,880. To determine more definitely the quantity of mathematical literature represented, two methods suggest themselves. One is to assume that the number of *pages* filled by those reviews is proportional, on the average, to the quantity of literature published each year; or, if it seems more reasonable, proportional to the number and importance of ideas developed. The other is to count the titles listed in each table of contents, and take those for a measure. Diagrams are here shown, giving the results of both methods, the years being set at equal intervals along a horizontal base, and their output in pages or in titles being set off upward. A broken line joins

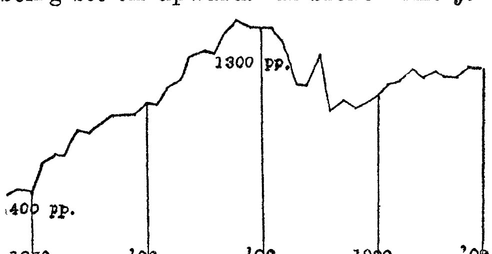


Fig. 1. *Jahrbuch über die Fortschritte der Mathematik.* Increase in size during forty years.

the points so marked; thus the area between graph and baseline, widening from decade to decade with one or two interruptions, conveys a fair idea of the growth in this kind of literature. One of the interruptions comes in the years 1893–4, where a double volume was issued in an attempt to overtake the flight of time. It shows pre-

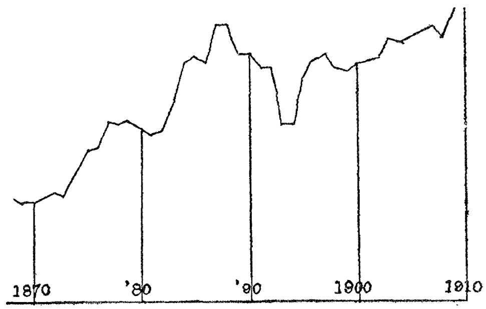


Fig. 2. Annual Number of Titles of Mathematical Articles and Books, 1868–1909.

sumably that considerable matter properly belonging in those years was pushed into the preceding and the following volume, swelling them unduly. The general impression

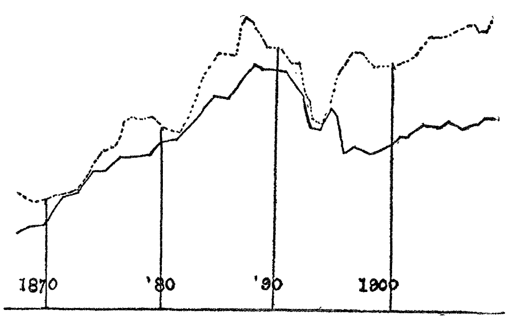


Fig. 3. Number of Pages; Number of Titles, for Comparison, in Dotted Curve.

from these two curves (if I may call them so) is that the two methods agree approximately, and that either one may be used as may be convenient. But we shall test this further on particular divisions of the science.

We see that the annual publication has doubled during the four decades, and considerably more; also that the space given to each review has decreased. Causes for the first are the increased attendance at universities, the general advance in zeal for research as a part of the great onward movement in natural science, and the increase in funds available to support publication of new results. For the second fact, the condensation of reviews, it is certain that much space is saved by the possibility of reference to earlier volumes where similar articles were reported; and it may be suspected that the haste to print has given us many articles of little importance; and further, it is increasingly necessary to save the time of the reader—Fechner's law may perhaps hold here, as in the ratio between nerve-stimuli and sensations: the referees are competent, busy workers in various departments of research, and fatigue effects must be looked for in their treatment of minor matters at least.

For convenience I separate the list of contents into five principal groups, each group embracing several chapters or sections; these will suffice at least for a first study. They are philosophy and biography, algebra, analysis, geometry and applied mathematics. To any specialist this seems too coarse a division for any definite result, but we are seeking general information. Our division *algebra* includes not only the theory of equations and their systems, but also such diverse subjects as Galois groups, theory of integers and other numbers, determinants, invariants of linear substitutions, probability and series. So too *analysis* covers a long range, from the elements of differential and integral calculus to the theory of real and complex functions of one or of several variables, general and particular. *Geometry* covers both the Euclidean and non-Euclidean, synthetic and analytic, the elementary and much that is so advanced as to be hardly distinguishable from algebra or from abstract logic. So *Mechanics,* all branches of mathematical physics, astronomy, geodesy and even meteorology are combined under the single rubric, *Applied Mathematics.*

There is a possibility of testing roughly those data, by comparison through the years 1891–1909 with the *Revue Semestrille,* a briefer report issued half-yearly and with less delay by the mathematical societies of the Netherlands, chiefly of Amsterdam. The comparison can not be quite satisfactory because the *Revue* classifies one article often under several rubrics, sometimes giving six or more references by different members for an article listed only once among the *Titles* in the *Jahrbuch.* Of course the tedious process of counting titles page by page through every volume is not to be thought of. Making this allowance, I have prepared with the help of Dr. Cowley, a collaborator on the staff of the *Revue,* graphs showing the numbers of

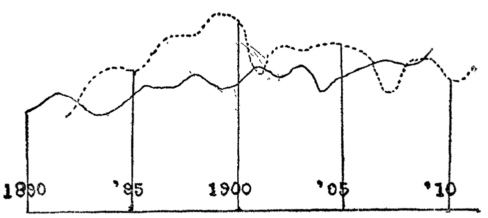


FIG. *A.* Algebra, Titles. *Jahrbuch,* Solid; *Revue Semestrielle,* Dotted Curve.

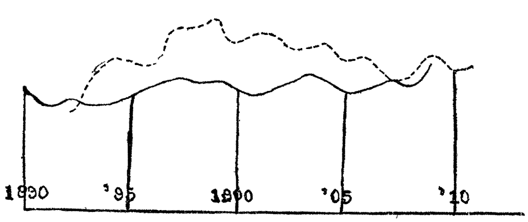


FIG. *B.* Analysis, Titles. *Jahrbuch,* Solid; *Revue Semestrielle,* Dotted Curve.

titles listed by both reports in the four topics, excluding philosophy and biography.

These may be considered satisfactory confirmation of the others, if we make two or three assumptions or observations. First, the scale adopted for comparison is probably too great on the *Jahrbuch* side. Titles here were measured in inches, and have

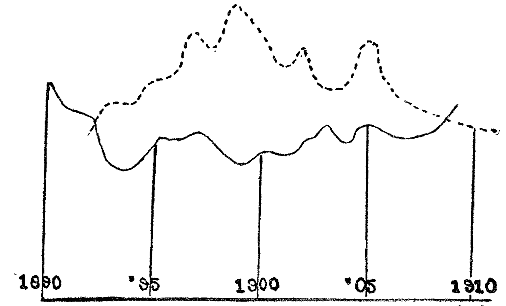


FIG. C. Geometry, Titles. *Jahrbuch,* Solid; *Revue Semestrielle,* Dotted Curve.

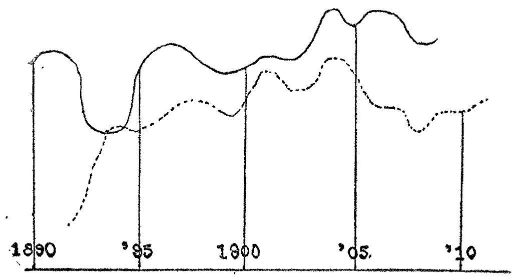


FIG. D. Applied Mathematics. *Jahrbuch,* Solid; *Revue Semestrielle,* Dotted Curve.

been plotted as containing 10 to the inch. By random samples it appears that 7 or 8 would have been nearly correct. Second: If by this correction the *Jahrbuch* curve were lowered relatively, still the divergence would be disproportionate for different years; less in earlier and later years, greater near the middle of the period. As to this, it is possible to surmise that the frequent change in the *personnel* of the staff, together with the fact that each one turns in reports on all the contents of his assigned periodicals rather than upon a list pertaining to his own few specialties, will explain much of the observed effect. In the first few years habits of work were not yet settled, and after several years it may easily have happened that cross-references of minor importance were found too numerous for utility and gradually reduced. Probably, thirdly, irregularities due to postponement or to lack of prompt arrival of publications would affect the *Revue* more markedly than the more leisurely *Jahrbuch.*

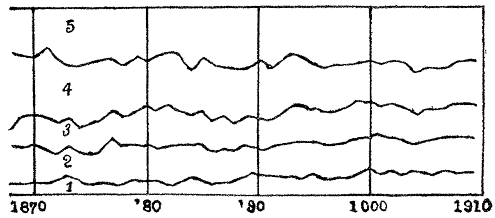
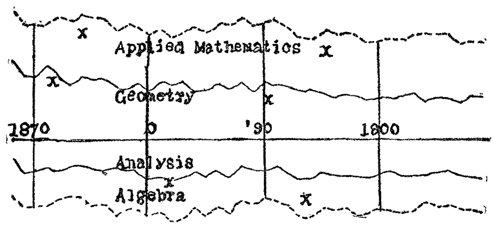


FIG. 4. Proportions of Titles in Large Divisions. (1) Philosophy, (2) Algebra, (3) Analysis, (4) Geometry, (5) Applied Mathematics.



FIGS. 5–6. Proportional Numbers of Titles in Four General Divisions of Mathematics. (*x*, Maxima.)

I show graphs of the proportionate amounts (*i. e.*, percentages) of the successive volumes, that fall into the several divisions; the first with boundaries straight and parallel, like covers of a book, the second with a straight line separating the abstract algebra and analysis on one side from the more concrete geometry and applied mathematics on the other. In this latter the outer boundaries are parallel curves, and their fluctuations enable us to trace the occasional shift of interest toward concrete or abstract.

On this diagram it is noticeable that the proportions of writings in the different divisions are by no means constant. Roughly estimated, the averages are: Applied mathematics 30 per cent., geometry 25 per cent., analysis 20 per cent., algebra 18 per cent. and philosophy, history, etc., together 7 per cent. But there are certain times of wide variation from the average. In physics, for example, there is a period of increase from 1883 to 1890, a time when Maxwell's "Electricity and Magnetism," Rowland's researches in the spectrum, Helmholtz's Faraday lecture, and certainly the influence of Thomson and Tait's great treatise on "Natural Philosophy," might help to explain the rapid growth of interest and discussion. The lesser, though well marked, influx of energy in the years following 1895 may be due in part to the discovery of radium and the new theories of electrons and atoms. Of course such surmises can only be verified or corrected by close examination of the subjects that are grouped together indiscriminately in this division.

Such fluctuations occur not only in these most concrete branches, but are discernible in all the others. Such variations in the choice of subjects for investigation one is tempted to call changes of fashion, so capricious or accidental do they appear to a superficial observer. On this same diagram, in geometry, note the disproportionate breadth of the stream through the '70's, and its slow narrowing thereafter. See in analysis, too, the breadth near 1880, and later, some expansion soon after 1890. Algebra enjoyed maximal periods not far from 1878, and has overflowed its usual boundaries again ever since 1899. The gradual augmentation in philosophy, history, biography might have been expected, but is also in part traceable to the influence of imitation and to vagaries in classification.

One may pick flaws in classification, but there are excuses, one of them in this very fact of changing fashions. For a set of categories that answer well enough in 1868 do not contain explicitly all the topics that interest even conservative mathematicians in 1880, still less in 1910. No long disquisition on this subject is needed, for nearly every one has some pet grievance against a much more elaborate arrangement, the Dewey system of classification of books. But also a part of the fault, in the case before us, lay in the imperfect office arrangements even more than in the system adopted. Against this source of error the more recent French system has guarded, by adopting numbers for a designating mark. Of course the *Jahrbuch*, published in Berlin, ignores this improvement, but it does from time to time add new titles to chapters and new subdivisions under them. My two complaints against the editors were first their lack of discrimination between two subjects called by the same name: theory of forms in the algebra of continuous variables, and theory of forms in the domain of whole numbers or integers. When looking for important memoirs on the one subject, one must always search with a careful eye the other chapter also. And in the second place they had made no place for any *Theory of Groups* except that concerned with permutations (hardly even for that until very recent years), and ignored completely the creation of a great new department of activity by *Sophus Lie* in 1870, under the same caption with a difference, viz., the "Theory of Continuous Groups."

In all this tabulation and construction of graphs, it has been assumed that one article is like another in importance, one idea as fertile as its rival, one name of an investigator no more to be noticed than any other. This assumption, however, is harmless, for it deceives nobody. As well try to convince a student of stars that all parts of the nightly sky are of equal brightness, that there are no luminaries of first magnitude and no Galaxy resplendent with its organized myriads! But in a first approximation we are at liberty to make errors, provided we do so with the consciousness that they are errors and that they call for subsequent discrimination and revision.

Look then at a few graphs relating to single divisions or subdivisions.

In 1868 and 1869 was published an epoch-making work, the "Geometry of Straight Lines," by Julius Plücker, professor of physics and mathematics at the University of Bonn. Unconsciously students of geometry had presumed that the space we live in must be conceived as built up of points, or minute bodies, having vanishing magnitude or dimension in all directions. Plücker pointed out that to the eye of pure mathematics it is just as true that space is built of *straight lines*. The intersection of the lines of geometry is no obstacle to such a theory, any more than if they were rays of starlight! With this new conception of space, amenable to a beautiful algebraic treatment, came a great stimulation of speculation and exploration in geometrical realms both old and new. *Plücker*, his brilliant pupil *Klein*, *Clifford* and *Cayley* are the centers of greatest energy in this movement, whose force persisted for twenty years and more. I show here a graph for geometry, no longer as a

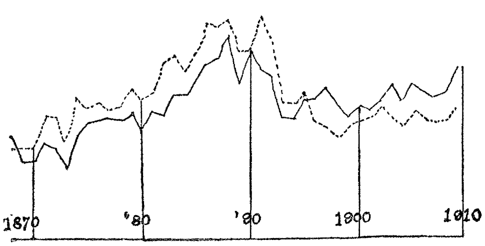


Fig. 7. Geometry, Number of Titles; Number of Pages, for Comparison, shown by Dotted Curve. Agreement, fair.

greater or less part of the entire range of mathematics, but measured, so to say, absolutely—the number of titles or of pages given to it in each annual volume of the *Jahrbuch*. Judged upon either basis, geometry seems to have doubled in rate of production from 1870 to 1890, then to have fallen off a third, to regain most of this loss after 1899.

In the universities of Germany the decline was decidedly more violent than this graph can show; for men trained in a specialty do not all change suddenly their interest or their line of study. So the new impulse, if it comes, will lag in its manifestation on this graph. Geometry was transplanted into Italy during the '80's, and the notion of *Group*, worked out in partial applications earlier by Klein, Sophus Lie, and their schools, was extended, energized and developed in larger geometric shape by young, ardent, brilliant sons of the men who in 1870 had created a new kingdom of Italy. A new society began to publish *Rendiconti* at Palermo ('88) and we note the point, about 1893, where this rising tide meets and overcomes the ebb of the German wave, Segre is here the great name, at first; Castelnuovo and Enriques soon rise to altitudes before unknown, algebraic *surfaces* now proving to be no less interesting than algebraic *curves* had been in the 70's and 80's.

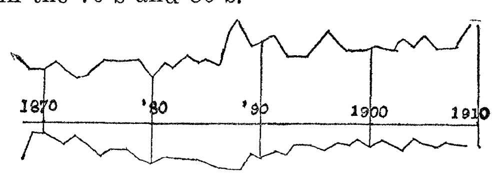


Fig. 8. Above, Analytic Geometry, Titles; Below, Modern Synthetic Geometry (same scale).

Let us further distinguish, dividing geometry into the pure or synthetic, and the algebraic. Both had experienced revival in Germany, the synthetic geometry through a combination of influences, conspicuously through the publication of *Reye's* lectures in form pedagogically perfect, with full illustration and touched with that scientific fire which is almost poetic inspiration. Our graph is arranged to show on opposite sides of a base-line, by numbers of titles, the output of analytic and algebraic geometry above, the synthetic below. The ordinates are absolute, not percentages; and further argument is unnecessary to establish the thesis that research is ruled partly by fashion; the maximum of synthetic work about 1887, and its decline through

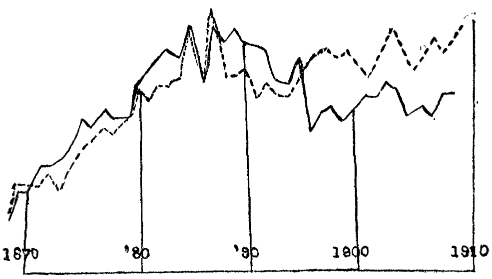


FIG. 9. Analysis, Pages; Titles, Dotted Curve. Both show a maximum about 1886, a minimum about 1895.

the following twenty years, is proof sufficient.

One graph shows the growth of what is called *analysis,* that great body of knowledge which takes its rise equally from calculus, from the algebra of imaginaries, from the intuitions and the critically refined developments of geometry, and from abstract logic: the common servant and chief ruler of the other branches of mathematics. The page-chart solid, the record of titles broken, both show a general increase in the amount of work, possibly a trebling in forty years. A first maximum appears before 1890, probably the culmination of waves

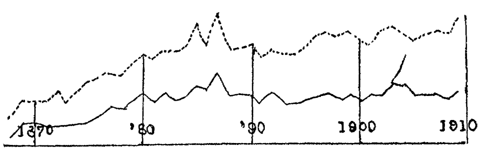


FIG. 10. Theory of Functions; Titles. Dotted Line: Analysis (Reduced).

set in motion by *Weierstrass* and *Fuchs* in Berlin, by *Riemann* in Göttingen, by *Hermite* in Paris, *Mittag-Leffler* in Stockholm, *Dini* and *Brioschi* in Italy. The great energy of these giant intellects was directed mainly to founding on a basis critically unassailable the theory of functions, before this time somewhat loose and uncertain, and to developing its particulars from general grounds, rather than by piecemeal as was the necessity earlier. So in the next graph we see how considerable a part of the growth of analysis prior to 1887 was due to activity in this—its central part, the theory of functions. But the fashion changes, and a new sweep of the curve upward does not occur until after 1900, when a new impulse comes from the theory of integral equations (not yet recognized as a separate field by the *Jahrbuch* editors) and from the influence of another master mind, *Hilbert* of Göttingen. In our own country too this branch of science is forwarded, notably at Harvard, Chicago and Yale.

One more example is perhaps the most striking of all. *Algebra* is shown first,

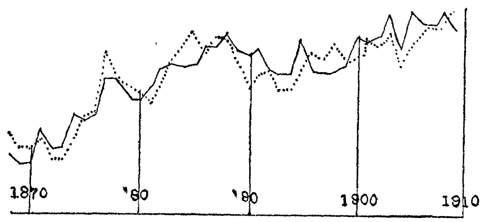


FIG. 11. Algebra, Including Series and Groups. Solid Line, Pages; Dotted Line, Titles.

pages on a solid curve, titles on a dotted one, quite nearly in accord. Each decade has its chief new subject for expansion, and growth is not far from steady. Equations, now up to the seventh and eighth degrees: determinants, invariants of linear groups of operations, and substitution groups all have had their years of plenty under this curve, and give it a strong, well-grown look. The part of this work that was shared most largely by our own countrymen is the theory of algebraic forms, quantics or invariants and covariants, as it is variously called. *Sylvester,* first professor of mathematics at Johns Hopkins University, had done fundamental work in this field in his youth, in the early 50's. At Baltimore the eager requests of students led him back to that line of research, and many joined him in producing extensive and valuable work; much of it published in the *American Journal of Mathematics,* which he founded. In Germany the same subject engaged attention, at the same time or a little earlier, through the efforts of *Aronhold* at Berlin—a true pioneer; of *Clebsch* at Göttingen and *Gordan* at Erlangen, the last two founding in 1869 a new journal, the *Mathematische Annalen,* to facilitate the publication needed for the work of themselves and their enlarging circle of progressives.

The rapid rise, the climax and the decline to a low normal, in this theory of forms or *invariants,* is shown in a graph which amply repays this rapid statistical study.

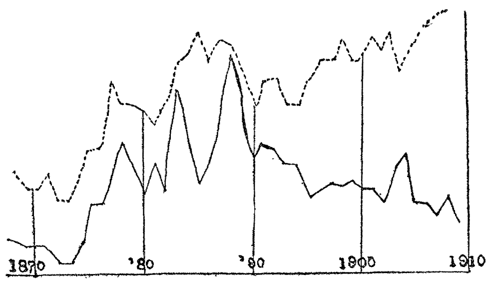


Fig. 12. Algebraic Forms, Invariants, etc.: Titles. Algebra, including Number-theory and Groups, Above (Reduced).

The solid curve shows the annual number of titles, while the dotted graph above, on smaller scale, shows how the general division, algebra, was fluctuating. Here the fashion reaches its acme before 1890, an increase twice as rapid as that of the main division; and declines most surprisingly. Sylvester returned to England in 1884; and in Germany a climactic series of discoveries by Hilbert set a temporary high mark, discouraging further effort for the time. But a later maximum, 1905, has in it a guarantee that growth is not impossible; this I do not pretend to explain, but the fact is obvious.

The field of differential equations has always held attraction for mathematicians, principally because of its close contact with physics and geometry. Its development naturally waited for that of the theory of functions. We see the researches in this

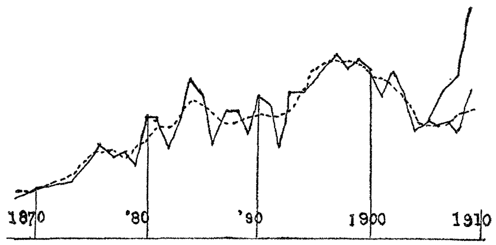


Fig. 13. Differential Equations: Titles. The dotted curve is averaged for three years.

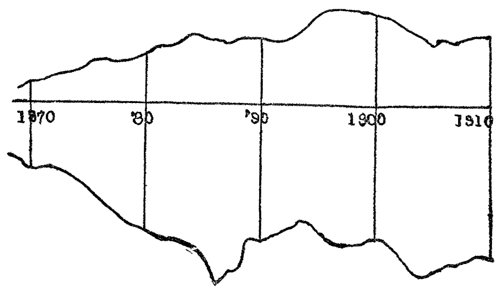


Fig. 14. Differential Equations Above; Below, the Residue of Analysis, Number of Titles.

department increasing in number slowly from 1870, under the combined influence of Weierstrass, Darboux and Lie; and note a slight decline about 1886, followed by a marked recovery and advance during the publication of lectures by Forsyth, Picard, Goursat and Painlevé. It is of interest to see the relative variation in differential equations on the one hand, and all the rest of analysis on the other.

Finally we examine in a separate diagram the fluctuation in absolute quantity of work on the mathematical theory of electricity and magnetism, and its ratio to the whole of applied mathematics. It remains less than one fourth of the whole, but rises after 1873

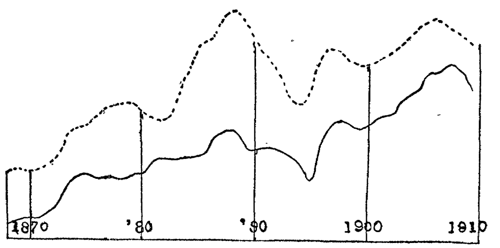


FIG. 15. Electricity and Magnetism; Dotted Line, Applied Mathematics (½ scale).

fairly steadily toward that fraction. It has been mentioned already that 1873 was the year of publication of Maxwell's "Electricity and Magnetism," and of Thomson and Tait's "Natural Philosophy." Maxwell's book appears in German translation in 1883. About 1880 Hertz began his remarkable experimental work verifying the theories of Faraday and Maxwell, with effects certainly contributory to the swelling of this tide. An apparent ebb near 1894 is explained in part by the increasing preoccupation of the referee in that department. Immediately after that date the increase of matter is resumed, partly under the new stimulus of Röntgen rays (1897) and of the electron theory (about 1902). By comparison, this special impulse seems not to be shared by the other departments of applied mathematics.

A careful inspection of these graphs might lead some specialists to lament the constant shifting of the center of mathematical interest. To students, however, who are about embarking on the sea of research, it may yield such profitable hints as the mariner draws from a chart of prevailing winds and currents. H. S. WHITE

VASSAR COLLEGE

NICHOLAI ALEXEYEVICH OUMOV

THE recent death of Professor Nicholai Alexeyevich Oumov deprived Russia and the world of one of those remarkable men, unfortunately rare in our age of sharply defined specialization, in whom the powers of analysis and synthesis were so well blended together, as to enable him to be more than simply a great physicist, or a great chemist, or a great cosmographer, but a great philosopher.

It seems that modesty is a usual attribute of greatness, and Oumov was a modest man, indeed. Perhaps this accounts for the fact that he is so very little known in this country and in England, although the continental Europe, especially Germany, knows him well and has a profound respect for his works. His biography, however, seems to be a matter of knowledge only within a small circle of his countrymen, and it seems desirable to supply this want.

Oumov was born in 1846. To his father, a physician by profession, he owes his profound and wide interest in physical science. At the age of twenty-three he was graduated by the faculty of physics and mathematics of the University of Moscow. After his graduation he entered the car-construction works of Williams and Buchteeff. Later on he registered at the St. Petersburg Technological Institute, but after two months' attendance, he received an offer to return to the University of Moscow and prepare for a professorship there.

In 1871, however, he received appointment as a privat-docent in physics at the University of Odessa, where, several years later, he was promoted to professorship. In 1893 he was transferred to the University of Moscow, where he lectured successively on general, mathematical and experimental physics.

In 1905 the Russian universities received a charter of autonomy, by which the administration of the institutions was left in the hands of the university council, while the student body received certain privileges of self-government. However, towards the end of 1910, when the late Kasso became the Russian minister of education, attempts were made to deprive the universities of their autonomy. Early in 1911 a strike of the Moscow students, provoked by the agents of the police, as was later determined, gave the government the desired opportunity, and police control over the