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GRADUATE MATHEMATICAL INSTRUCTION FOR GRADUATE STUDENTS NOT INTENDING TO BECOME MATHEMATICIANS[1]

IN his "Annual Report" under date of November last, the President of Columbia University speaks in vigorous terms of what he believes to be the increasing failure of present-day advanced instruction to fulfil one of the chief purposes for which institutions of higher learning are established and maintained.

President Butler, in the course of an interesting section devoted to college and university teaching, says:

> A matter that is closely related to poor teaching is found in the growing tendency of colleges and universities to vocationalize all their instruction. A given department will plan all its courses of instruction solely from the point of view of the student who is going to specialize in that field. It is increasingly difficult for those who have the very proper desire to gain some real knowledge of a given topic without intending to become specialists in it. A university department is not well organized and is not doing its duty until it establishes and maintains at least one strong substantial university course designed primarily for students of maturity and power, which course will be an end in itself and will present to those who take it a general view of the subject-matter of a designated field of knowledge, its methods, its literature and its results. It should be possible for an advanced student specializing in some other field to gain a general knowledge of physical problems and processes without becoming a physicist; or a general knowledge of chemical problems and processes without becoming a chemist; or a general knowledge of zoological problems and processes without becoming a zoologist; or a general

[1] An address delivered before Section A of the American Association for the Advancement of Science, December 30, 1914.

knowledge of mathematical problems and processes without becoming a mathematician.

This is a large matter, involving all the cardinal divisions of knowledge. I have neither time nor competence to deal with it fully or explicitly in all its bearings. As indicated by the title of this address it is my intention to confine myself, not indeed exclusively but in the main, to consideration of the question in its relation to advanced instruction in mathematics. The obvious advantages of this restriction will not, I believe, be counterbalanced by equal disadvantages. For, much as the principal subjects of university instruction differ among themselves, it is yet true that as instruments of education they have a common character and for their efficacy as such depend fundamentally upon the same educational principles. A discussion, therefore, of an important and representative part of the general question will naturally derive no little of whatever interest and value it may have from its implicit bearing upon the whole. It is not indeed my intention to depend solely upon such implicit bearings nor upon the representative character of mathematics to intimate my opinion respecting the question in its relation to other subjects. On the contrary, I am going to assume that specialists in other fields will allow me, as a lay neighbor fairly inclined to minding his own affairs, the privilege of some quite explicit preliminary remarks upon the larger question.

I suspect that my interest in the matter is in a measure temperamental; and my conviction in the premises, though it is not, I believe, an unreasoned one, may be somewhat colored by inborn predilection. At all events I own that a good many years of devotion to one field of knowledge has not destroyed in me a certain fondness for avocational studies, for books that deal with large subjects in large ways, and for men who, uniting the generalist with the specialist in a single gigantic personality, can show you perspectives, contours and reliefs, a great subject or a great doctrine in its principal aspects, in its continental bearings, without first compelling you to survey it pebble by pebble and inch by inch. I can not remember the time when it did not seem to me to be the very first obligation of universities to cherish instruction of the kind that is given and received in the avocational as distinguished from the vocational spirit—the kind of instruction that has for its aim, not action but understanding, not utilities but ideas, not efficiency but enlightenment, not prosperity but magnanimity. For without intelligence and magnanimity—without light and soul—no form of being can be noble and every species of conduct is but a kind of blundering in the night. I could hardly say more explicitly that I agree heartily and entirely with the main contention of President Butler's pronouncement. Indeed I should go a step further than he has gone. He has said that a university *department* is not well organized and is not doing its duty until it establishes and maintains the kind of instruction I have tried to characterize. To that statement I venture to add explicitly—what is of course implicit in it—that a *university* is not well organized and is not doing *its* duty until it makes provision whereby the various departments are enabled to foster the kind of instruction we are talking about. That in all major subjects of university instruction there ought to be given courses designed for students of "maturity and power" who, whilst specializing in one subject or one field, desire to generalize in others, appears to me to be from every point of view so reasonable and just a proposition that it would not occur to me to regard it as questionable or debatable were it not for the fact that it actually is questioned and debated by teachers of eminence and authority.

What is there in the contention about which men may differ? Dr. Butler has said that there is a "growing tendency of college and university departments to vocationalize all their instruction." Is the statement erroneous? It may, I think, be questioned whether the tendency is growing. I hope it is not. Of course specialization is not a new thing in the world. It is far older than history. Let it be granted that it is here to stay, for it is indispensable to the advancement of knowledge and to the conduct of human affairs. Every one knows that. There is, however, some evidence that specialization is becoming, indeed that it has become, wiser, less exclusive, more temperate. The symptoms of what not long ago promised to become a kind of specialism mania appear to be somewhat less pronounced. Recognition of the fact that specialization is in constant peril of becoming so minute and narrow as to defeat its own ends is now a commonplace among specialists themselves, many of whom have learned the lesson through sad experience, others from observation. Specialists are discoverers. One of our recent discoveries is the discovery of a very old truth: we have discovered that no work can be really great which does not contain some element or touch of the universal, and that is not exactly a new insight. Leonardo da Vinci says:

> We may frankly admit that certain people deceive themselves who apply the title "a good master" to a painter who can only do the head or the figure well. Surely it is no great achievement if by studying one thing only during his whole lifetime he attain to some degree of excellence therein!

The conviction seems to be gaining ground that in the republic of learning the ideal citizen is neither the ignorant specialist, however profound he may be, nor the shallow generalist, however wide the range of his interest and enlightenment. It is not important, however, in this connection to ascertain whether the vocationalizing tendency is at present increasing or decreasing or stationary. What is important is to recognize the fact that the tendency, be it waxing or waning, actually exists, and that it operates in such strength as practically to exclude all provision for the student who, if I may so express it, would qualify himself to gaze into the heavens intelligently without having to pursue courses designed for none but such as would emulate a Newton or a Laplace. If any one doubts that such is the actual state of the case, the remedy is very simple: let him choose at random a dozen or a score of the principal universities and examine their bulletins of instruction in the major fields of knowledge.

Another element—an extremely important element—of President Butler's contention is present in the form of a double assumption: it is assumed that in any university community there are serious and capable students whose primary aim is indeed the winning of mastery in a chosen field of knowledge but who at the same time desire to gain some understanding of other fields—some intelligence of their enterprises, their genius, their methods and their achievements; it is further assumed that this non-vocational or avocational propensity is legitimate and laudable. Are the assumptions correct? The latter one involves a question of values and will be dealt with presently. In respect of the former we have to do with what mathematicians call an existence theorem: Do the students described exist? They do. Can the fact be demonstrated—deductively proved? It can not. How, then, may we know it to be true? The answer is: partly by observation, partly by experience, partly by inference and partly by being candid with ourselves. Who is there among us that is unwilling to admit that he himself now is or at least once was a student of the kind?

Where is the university professor to whom such students have not revealed themselves as such in conversation? Who is it that has not learned of their existence through the testimony of others? No doubt some of us not only have known students of the kind, but have tried in a measure to serve them. We may as well be frank. I have myself for some years offered in my subject a course designed in large part for students having no vocational interest in mathematics. I may be permitted to say, for what the testimony may be worth, that the response has been good. The attendance has been composed about equally of students who were not looking forward to a career in mathematics and of students who were. And this leads me to say, in passing, that, if the latter students were asked to explain what value such instruction could have for them, they would probably answer that it served to give them some knowledge *about* a great subject which they could hardly hope to acquire from courses designed solely to give knowledge *of* the subject. Every one knows that it often is of great advantage to treat a subject as an object. One of the chief values of n-dimensional geometry is that it enables us to contemplate ordinary space from the outside, as even those who have but little imagination can contemplate a plane because it does not immerse them. Returning from this digression, permit me to ask: if, without trying to discover the type of student in question, we yet become aware, quite casually, that the type actually exists, is it not legitimate to infer that it is much more numerously represented than is commonly supposed? And if such students occasionally make their presence known even when we do not offer them the kind of instruction to render their wants articulate, is it not reasonable to infer that the provision of such instruction would have the effect of revealing them in much greater numbers?

Indeed it does not seem unreasonable to suppose that a "strong substantial course" of the kind in question, in whatever great subject it were given, would be attended not only by considerable numbers of regular students but in a measure also by officers of instruction in other subjects and even perhaps by other qualified residents of an academic community. Only the other day one of my mathematical colleagues said to me that he would rejoice in an opportunity to attend such a course in physics. The dean of a great school of law not long ago expressed the wish that some one might write a book on mathematics in such a way as would enable students like himself to learn something of the innerness of this science, something of its spirit, its range, its ways, achievements and aspiration. I have known an eminent professor of economics to join a beginners' class in analytical geometry. Very recently one of the major prophets of philosophy declared it to be his intention to suspend for a season his own special activity in order to devote himself to acquiring some knowledge of modern mathematics. Similar instances abound and might be cited by any one not only at great length, but in connection with every cardinal division of knowledge. Their significance is plain. They are but additional tokens of the fact that the race of catholic-minded men has not been extinguished by the reigning specialism of the time, but that among students and scholars there are still to be found those whose curiosity and intellectual interests surpass all professional limits and crave instruction more generic in kind, more liberal, if you please, and ampler in its scope, than our vocationalized programs afford.

As to the question of values, I maintain that the desire of such men is entirely legitimate, that it is wholesome and praiseworthy, that it deserves to be stimulated, and that universities ought to meet it, if they can. Indeed, all this seems to me so obvious that I find it a little difficult to treat it seriously as a question. If the matter must be debated, let it be debated on worthy ground. To say, as proponents sometimes say, that, inasmuch as all knowledge turns out sooner or later to be useful, students preparing for a given vocation by specializing in a given field may profitably seek some general acquaintance with other fields *because* such general knowledge will indirectly increase their vocational equipment, is to offer a consideration which, though in itself it is just enough, yet degrades the discussion from its appropriate level, which is that of an ideal humanity, down to the level of mere efficiency and practicianism. No doubt one engaged in minutely studying the topography of a given locality because he intends to reside in it might be plausibly advised to study also the general geography of the globe on the ground that his special topographical knowledge would be thus enhanced, and that, moreover, he might some time desire to travel. But if we ventured to counsel him so, he might reply: What you say is true. But why do you ply me with such low considerations? Why do you regard me as something crawling on its belly? Don't you know that I ought to acquire a general knowledge of geography, not primarily because it may be useful to me as a resident here or as a possible traveler, but because such knowledge is essential to me in my character as a man? The rebuke, if we were fortunately capable of feeling it, would be well deserved. A man building a bridge is greater than the engineer; a man planting seed is greater than the farmer; a man teaching calculus is greater than the mathematican; a man presiding at a faculty meeting is greater than the dean or the president. We may as well remember that man is superior to any of his occupations. His supreme vocation is not law or medicine or theology or commerce or war or journalism or chemistry or physics or mathematics or literature or any specific science or art or activity; it is intelligence, and it is this supreme vocation of man as man that gives to universities their supreme obligation. It is unworthy of a university to conceive of man as if he were created to be the servant of utilities, trades, professions and careers; these things are for *him*: not ends but means. It is said that intelligence is good because it prospers us in our trades, industries and professions; it ought to be said that these things are good because and in so far as they prosper intelligence. Even if we do not conceive the office of intelligence to be that of contributing to being in its highest form, which consists in understanding, even if we conceive its function less nobly as that of enabling us to adjust ourselves to our environment, the same conclusion holds. For what is our environment? Is it wholly or mainly a matter of sensible circumstance—sea and land and sky, heat and cold, day and night, seasons, food, raiment, and the like? Far from it. It is rather a matter of spiritual circumstances—ideas, sentiments, doctrines, sciences, institutions, and arts. It is in respect of this ever-changing and ever-developing world of spiritual things, it is in respect of this invisible and intangible environment of life, that universities, whilst aiming to give mastery in this part or that, are at the same time under equal obligation to give to such as can receive it some general orientation in the whole.

And now as to the question of feasibility. Can the thing be done? So far as mathematics is concerned I am confident that it can, and I have a strong lay suspicion that it can be done in all other subjects.

It is my main purpose to show, with some regard to concreteness and detail, that the thing is feasible in mathematics. Before doing so, however, I desire to view the matter a little further in its general aspect and in particular to deal with some of the considerations that tend to deter many scientific specialists from entering upon the enterprise.

One of the considerations, and one, too, that is often but little understood, and so leads to wrong imputations of motive, though it is in a sense distinctly creditable to those who are influenced by it, is the consideration that relates to intricacy and technicality of subject-matter and doctrine. Every specialist knows that the principal developments in his branch of science are too intricate, too technical and too remote from the threshold of the matter to be accessible to laymen, whatever their abilities and attainments in foreign fields. Not only does he know that there is thus but relatively little of his science which laymen can understand but he knows also that the portions which they can not understand are in general precisely those of greatest interest and beauty. And knowing this, he feels, sometimes very strongly, that were he to endeavor by means of a lecture course to give laymen a general acquaintance with his subject, he could not fail to incur the guilt of giving them, not merely an inadequate impression, but an essentially false impression, of the nature, significance and dignity of a great field of knowledge. His hesitance therefore, is not due, as it is sometimes thought to be, to indifference or to selfishness. Rather is it due to a sense of loyalty to truth, to a sense of veracity, to an unwillingness to mislead or deceive. Of course strange things do sometimes happen, and it is barely conceivable that once in a long time nature may, in a sportive mood, produce a kind of specialist whose subject affects him much as the possession of an apple or a piece of candy affects the boy who goes round the corner in order to have it all himself. But if the type exist, not many men could claim the odd distinction of belonging to it. Specialists are as generous and humane as other men. Their subjects affect them as that same boy is affected when, if he chance to come suddenly upon some strange kind of flower or bird, he at once summons his sister or brother or father or mother or other friend to share in his surprise and joy. There is this difference, however—the specialist must, unfortunately, suffer *his* joy in solitude unless and until he finds a comrade in kind. I admit that the deterrent consideration in question is thoroughly intelligible. I contend that the motive it involves presents an attractive aspect. But I can not think it of sufficient weight to be decisive. It involves, I believe, an erroneous estimate of values, a fallacious view of the ways of truth to men. A few years ago, when making a railway journey through one of the most imposing parts of the Rocky Mountains, I was tempted like many another passenger to procure some photographs of the scenery in order to convey to far-away friends some notion of the wonders of it. So far, however, did the actual scenery surpass the pictures of it, excellent as these were, that I decided not to buy them, feeling it were better to convey no impression at all than to give one so inferior to my own. No doubt the decision might be defended on the ground of its motive. Did it not originate in a certain laudable sense of obligation to truth? Nevertheless, as I am now convinced, the decision was silly. For in accordance with the same principle it is plain that I ought to have wished to have my own impressions erased, seeing that they must have been quite as inferior to those of a widely experienced mountaineer as those which the pictures could have given were inferior to mine. Who is so foolish as to argue that no one should learn anything about, say London, unless he means to master all its plans, its architecture and its history in their every phase, feature and detail? Who would contend that, because we are permitted to know only so little of what is happening in the European war, we ought to remain in total ignorance of it? Who would say that no one may with propriety seek to learn something about ancient Rome unless he is bent on becoming a Gibbon or a Mommsen? It is undoubtedly true that an endeavor to present a body of doctrine or a science to such as can not receive it fully must result in giving a false impression of the truth. But the notion that such an endeavor is therefore wrong is a notion which, if consistently and thoroughly carried out, would put the human mind entirely out of commission. All impressions, all views, all theories, all doctrines, all sciences are false in the sense of being partial, imperfect, incomplete. "Il n'y a plus des problèmes résolus et d'autres qui ne le sont pas, il y a seulement des problèmes *plus ou moins* résolus," said Henri Poincaré. Every one must see that, but for the helpfulness of views which because incomplete are also in a measure false, even the practical conduct of life, not to say the advancement of science, would be impossible. There is no other choice: either we must subsist upon fragments or perish.

Again, many a specialist shrinks from trying to present his subject to laymen because he looks upon such activity as a species of what is called popularization of science, and he believes that such popularization, even in its best sense, closely resembles vulgarization in its worst. He fancies that there is a sharp line bounding off knowledge that is mere knowledge from knowledge that is scientific. In his view science is for specialists and for specialists only. He declines, on something like moral and esthetic grounds, to engage in what he calls playing to the gallery. It might, of course, be said that there is more than one way of playing to the gallery. It could be said that one way consists in acting the rôle of one who imagines that his intellectual interests are so austere and elevated and his thought so profound that a just sense of the awful dignity of his vocation imposes upon him, when in presence of the vulgar multitude, the solemn law of silence. It would be ungenerous, however, if not unfair, to insist upon the justice of such a possible retort. Rather let it be granted, for it is true, that much so-called popularization of science *is* vicious, relieving the ignorant of their modesty without relieving them of their ignorance, equipping them with the vocabulary of knowledge without its content and so fostering not only a vain and empty conceit, but a certain facility of speech that is seemly, impressive and valuable only when, as is too seldom the case, it is accompanied by solid attainments. To say this, however, is not to lay an indictment against that kind of scientific popularization which was so happily illustrated by the very greatest men of antiquity, which was not disdained even by Galileo in the beginnings of modern science nor by Leonardo da Vinci, and which in our own time has engaged the interest and skill of such men as Clifford and Helmholtz, Haeckel and Huxley, Mach, Ostwald, Enriques and Henri Poincaré. It is not to arraign that variety of popularization which any one may behold in the constant movement of ideas, once reserved exclusively for graduate students, down into undergraduate curricula and which has, for example, made the doctrine of limits, analytical geometry, projective geometry, and the notions of the derivative and the integral available for presentation to college freshmen or even to high-school pupils. It is not to condemn that kind of popularization which is so natural a process that it actually goes on in a thousand ways all about us without our deliberate cooperation, without our intention or our consent, and has enriched the common sense and common knowledge of our time with countless precious elements from among the scientific and philosophic discoveries made by other generations of men.

Finally it remains to mention the important type of specialist in whom strongly predominates the predilection for research as distinguished from exposition. He knows, as every one knows, that through what is called practical applications of science many a scientific discovery is made to serve innumerable human beings who do not understand it and innumerable others who never can. He may or may not believe in avocational instruction; he may or may not regard intelligence as an ultimate good and an end in itself; he may or may not think that the arts and agencies for the dissemination of knowledge, as distinguished from the discovery and practical applications of truth, are important; he may or may not know that the art and the gifts of the great expositor are as important and as rare as those of the great investigator and less often owe their success to the favor of accident or chance. He may not even have seriously considered these things. He does know his own predilection; and so strong is his inclination towards research that for *him* to engage in exposition, espepecially in popular exposition, in avocational instruction for laymen, would be to sin against the authority of his vocation. This man, if he have intellectual powers fairly corresponding to the seeming authority and urgence of his inner call, belongs to a class whose rights are peculiarly sacred and whose freedom must be guarded in the interest of all mankind. It is not contended that every representative of a given subject is under obligation to expound it for the avocational interest and enlightenment of laymen. The contention is that such exposition is so important a service that any university department should contain at least one man who is at once willing and qualified to render it.

I come now to the keeping of my promise. It is to be shown that the service is practicable in the subject of mathematics and how it is so. Let us get clearly in mind the kind of persons for whom the instruction is to be primarily designed. They are to be students of "maturity and power"; they do not intend to become teachers, much less producers, of mathematics; they are probably specializing in other fields; they do not aim at becoming mathematicians; their interest in mathematics is not vocational, it is avocational; it is the interest of those whose curiosity transcends the limits of any specific profession or any specific form or field of activity; each of them knows that, whatever his own field may be, it is penetrated, overarched, compassed about by an infinitely vaster world of human interests and human achievements; they feel its immense presence, the poignant challenge of it all; as specialists they will win mastery over a little part, but they have heard the call to intelligence and are seeking orientation in the whole; this they know is a thing of mind; they are aware that the essential environment of a scholar's life is a spiritual environment—the invisible and intangible world of ideas, doctrines, institutions, sciences and arts; they know or they suspect that one of the great components of that world is mathematics; and so, not as candidates for a profession or a degree, but in their higher capacity as men and women, they desire to learn something of this science viewed as a human enterprise, as a body of human achievements; and they are willing to pay the price; they are not seeking entertainment, they are prepared to work—to listen, to read and to think.

And now we must ask: What measure of mathematical training is to be required of them as a preparation? In view of what has just been said it is evident that such training is not to be the whole of their equipment nor even the principal part of it, but it is an indispensable part. And the question is: How much mathematical knowledge and mathematical discipline is to be demanded? I have no desire to minimize my present task. I, therefore, propose that only so much mathematical preparation shall be demanded as can be gained in a year of collegiate study. Most of them will, of course, have had more; but I propose as a hypothesis that the amount named be regarded as an adequate minimum. But it does not include the differential and integral calculus. And is it not preposterous to talk of offering graduate instruction in mathematics to students who have not had a first course in the calculus? I am far from thinking so. A little reflection will suffice to show that in the case of such students as I have described it is very far from preposterous. In my opinion the absurdity would rather lie in demanding the calculus of them. No one is so foolish as to contend that a first course in the calculus is a *sufficient* preparation for undertaking the pursuit of graduate mathematical study. But to suppose it necessary is just as foolish as to suppose it sufficient. There was a time when it *was* necessary, and the belief that it is necessary now owes its persistence and currency to the inertia then acquired. Formerly it was necessary, because formerly all advanced courses, at least all initial courses of the kind, were either prolongations of the calculus, like differential equations, for example, or else courses in which the calculus played an essential instrumental rôle as in rational mechanics, or the usual introductions to function theory or to higher geometry or algebra. But, as every mathematician knows, that time has passed. It is true that courses for which a preliminary training in the calculus is essential still constitute and will continue to constitute the major part of the graduate offer of any department of mathematics. And quite apart from that consideration, it seems wise, in the case of intending graduate students who purpose to specialize in mathematics, to enforce the usual calculus requirement as affording some slight protection against immaturity and the lack of seriousness. But every mathematician knows that it is now practicable to provide a large and diversified body of genuinely graduate mathematical instruction for which the calculus is strictly not prerequisite.

Fortunately it is just the material that is thus available which is in itself best suited for the avocational instruction we are contemplating. As the calculus is not to be presupposed it goes without saying that this subject must find a place in the scheme. For evidently an advanced mathematical course devised and conducted in the interest of general intelligence can not be silent respecting "the most powerful weapon of thought yet devised by the wit of man." Technique is not sought and can not be given. The subject is not to be presented as to undergraduates. For the most part these gain facility with but little comprehension. It is to be presented to mature and capable students who seek, not facility, but understanding. Their desire is to acquire a general conception of the nature of the calculus and of its place in science and the history of thought—such a conception as will at least enable them as educated men to mention the subject without a feeling of sham or to hear it mentioned without a feeling of shame. A few well-considered lectures should suffice. At all events it would not require many to show the historical background of the calculus, to explain the nascence and nature of the scientific exigencies that gave it birth, to make clear the concepts of derivative and integral as the two central notions of its two great branches, and to present a few simple applications of these notions to intelligible problems of typical significance. Even the idea of a differential equation could be quickly reached, the nature of a solution explained, and simple examples given of physical and geometric interpretations. As to the range and power of the calculus, a sense and insight can be given, in some measure of course by a reference to its literature, but much more effectively by a few problems carefully selected from various fields of science and skillfully explained with a view to showing wherein the methods of the calculus are demanded and how they serve. Is not all this elementary and undergraduate? In point of nomenclature, yes. It is not necessary, however, to let words deceive us. We teach whole numbers to young children, but even Weierstrass was not aware of the logico-mathematical deeps that underlie cardinal arithmetic.

The calculus, however, is hardly the topic with which the course would naturally begin. A principal aim of the course should be to show what mathematics, in its inner nature, is—to lay bare its distinctive character. Its distinctive character, its structural nature, is that of a "hypothetico-deductive" system. Probably, therefore, it would be well to begin with an exposition of the nature and function of postulate systems and of the great rôle such systems have always played in the science, especially in the illustrious period of Greek mathematics and even more consciously and elaborately in our own time. It is plain that such an exposition can be made to yield fundamental insight into many matters of interest and importance not only in mathematics, but in logic, in psychology, in philosophy, and in the methodology of natural science and general thought. The material is almost superabundant, so numerous are the postulate systems that have been devised as foundations for many different branches of geometry, algebra, analysis, *Mengenlehre* and logic. A general survey of these, were it desirable to pass them all in review, would not be sufficient. It will be necessary to select a few systems of typical importance for minute examination with reference to such capital points as convenience, simplicity, adequacy, independence, compatibility and categoricalness. The necessity and presence of undefined terms in any and all systems will afford a suitable opportunity to deal with the highly important, much neglected and little understood subject of definition, its nature, varieties and function, in light of the recent literature, especially the suggestive handling of the matter by Enriques in his "Problems of Science." A given system once thus examined, the easy deduction of a few theorems will suffice to show the possibility and the process of erecting upon it a perfectly determinate and often imposing superstructure. And so will arise clearly the just conception of a mathematical doctrine as a body of thought composed of a few undefined together with many defined ideas and a few primitive or postulated propositions with many demonstrated ones, all concatenated and welded into a form independent of will and temporal vicissitudes. Revelation of the charm of the science will have been begun. A new revelation will result when next the possibility is shown of so interchanging undefined with defined ideas and postulates with demonstrated propositions that, despite such interchange of basal with superstructural elements, the doctrine as an autonomous whole will remain absolutely unchanged. But this is not all nor nearly all. It is only the beginning of what may be made a veritable apocalypse. Of great interest to any intellectual man or woman, of very great interest to students of logic, psychology, or philosophy, should be the light which it will be possible in this connection to throw upon the economic rôle of logic and upon the constitution of mind or the world of thought. I refer especially to the recently discovered fact that in interpreting a system of postulates we are not restricted to a single possibility, but that, on the contrary, such a system admits in general of a literally endless variety of interpretations; which means, for such is the make-up of our *Gedankenwelt*, that an infinitude of doctrines, widely different in respect of their psychological character and interest, have nevertheless a common form, being isomorphic, as we say, logically one, though spiritually many, reposing on a single base. And how foolish the instructor would be not to avail himself of the opportunity of showing, too, in the same connection, how various mathematical doctrines that differ not only psychologically, but logically also, are yet such that, by virtue of a partial agreement in their bases, they intersect one another, owning part of their content jointly, whilst being, in respect of the rest, mutually exclusive and incompatible. If, for example, it be some Euclidean system that he has been expounding, he will be able readily to show upon how seemingly slight changes of base there arise now this or that variety of non-Euclidean geometry, now a projective or an inversion geometry or some species or form of higher dimensionality. I need not say that analogous phenomena will in like manner present themselves in other mathematical fields. And it is of course obvious that as various doctrines are thus made to pass along in deliberate panorama it will be feasible to point out some of their salient and distinctive features, to indicate their historic settings, and to cite the more accessible portions of their respective literatures. Naturally in this connection and in the atmosphere of such a course the question will arise as to why it is that, or wherein, the hypothetico-deductive method fails of universal applicability. So there will be opportunity to teach the great lesson that this method is not rudimentary, but is an ideal, the ideal of intellect and science; to teach that mathematics is but the name of its occasional realization; and that, though the ideal is, relatively speaking, but seldom attained, yet its lure is universal, manifesting itself in the most widely differing domains, in the physical and mechanical assumptions of Newton, in the ethical postulates of Spinoza, in our federal constitution, even in the ten commandments, in every field where men have sought a body of principles to serve them as a basis of doctrine, conduct or achievement. And if it shall thus appear that mathematics is very high-placed as being, in respect of its method and its form, the ideal and the lure of thought in general, the fault must be imputed, not to the instructor, but to the nature of things.

In all this study of the postulational method the impression will be gained that the science of mathematics consists of a large and increasing number of more or less independent, somewhat closely related and often interpenetrating branches, constituting, not a jungle, but rather an immense, diversified, beautifully ordered forest; and that impression is just. At the same time another impression will be gained, namely, that the various branches rest, each of them, upon a foundation of its own. This impression will have to be corrected. It will have to be shown that the branch-foundations are not really fundamental in the science but are, literally and genuinely, component parts of the superstructure. It will have to be shown that mathematics as a whole, as a single unitary body of doctrine, rests upon a basis of primitive ideas and primitive propositions that lie far below the so-called branch-foundations and, in supporting the whole, support these as parts. The course will, therefore, turn to the task of acquainting its students with those strictly fundamental researches which we associate with such names as C. S. Peirce, Schroeder, Peano, Frege, Russell, Whitehead and others, and which have resulted in building underneath the traditional science a logico-mathematical sub-structure that is, philosophically, the most important of modern mathematical developments.

It must not be supposed, however, that the instruction must needs be, nor that it should preferably be, confined to questions of postulate and foundation, and I will devote the remainder of the time at my disposal to indicating briefly how, as it seems to me, a large or even a major part of the course may concern itself with matters more traditional and more concrete.

Any one can see that there is an abundance of available material. There is, for example, the history and significance of the great concept of function, a concept which mathematics has but slowly extracted and gradually refined from out the common content and experience of all minds and which on that account can be not only defined precisely and intelligibly to such laymen as are here concerned, but can also be clarified in many of its forms by means of manifold examples drawn from elementary mathematics, from the elements of other sciences, and from the most familiar phenomena of the work-a-day world.

Another available topic is the nature and rôle of the sovereign notion of limit. This, too, as every mathematician knows, admits of countless illustration and application within the radius of mathematical knowledge here presupposed. In this connection the structure and importance of what Sylvester called "the Grand Continuum," which so many scientific and other folk talk about unintelligently, will offer itself for explanation. And if the class fortunately contain students of philosophic mind, they will be edified and a little astonished perhaps when they are led to see that the method and the concept of limits are but mathematicized forms of a process and notion familiar in all domains of spiritual activity and known as idealization. Not improbably some of the students will be sufficiently enterprising to trace the mentioned similitude in some of its manifestations in natural science, in psychology, in philosophy, in jurisprudence, in literature and in art.

I have not mentioned the modern doctrine variously known as *Mengenlehre*, *Mannigfaltigkeitslehre*, the theory of point-sets, assemblages, manifolds or aggregates: a live and growing doctrine in which expert and layman are about equally interested and which, like a subtle and illuminating ether, is more and more pervading mathematics in all its branches. For the avocational instruction of lay students of "maturity and power" how rich a body of material is here, with all its fascinating distinctions of discrete and continuous, finite and infinite, denumerable and non-denumerable, orderless, ordered, and well-ordered, and with its teeming host of nearlying propositions, so interesting, so illuminating, often so amazing.

Finally, but far from exhausting the list, it remains to mention the great subjects of invariants and groups. Both of them admit of definition perfectly intelligible to disciplined laymen; both admit of endless elementary illustration, of having their mutual relations simply exemplified, of being shown in historic perspective, and of being strikingly connected, especially the notion of invariance, with the dominant enterprise of man: his ceaseless quest for the changeless amid the turmoil and transformation of the cosmic flux.

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PRELIMINARY REPORT ON A SHALER MEMORIAL STUDY OF CORAL REEFS

A LIBERAL grant from the Shaler Memorial Fund of Harvard University, supplemented by a generous subsidy from the British Association for the Advancement of Science with an invitation to attend its meeting in Australia last August as a foreign guest, enabled me to spend the greater part of the year 1914 in visiting a number of islands in the Pacific Ocean with the object of testing various theories that have been invented to account for coral reefs. Thirty-five islands, namely, Oahu in Hawaii, eighteen of the Fiji group, New Caledonia of which the entire coast line was traced, the three Loyalty islands, five of the New Hebrides, Rarotonga in the Cook group, and six of the Society islands, as well as a long stretch of the Queensland coast inside of the Great Barrier reef of northeastern Australia, were examined in greater or less detail. A brief statement of my results has been published in the *Proceedings of the National Academy of Sciences* for March, 1915. A full report will appear later, probably in the *Bulletin of the Museum of Comparative Zoology* at Harvard College. The general conclusions reached are here briefly summarized.

Any one of the eight or nine theories of coral reefs will satisfactorily account for the visible features of sea-level reefs themselves, provided the postulated conditions and processes of the invisible past are accepted: hence a study of the visible features of the reefs alone can not lead to any valid conclusion. Some independent witnesses must be interrogated, in the hope of detecting the true theory. The only witnesses, apart from sections obtained by deep and expensive borings, available for sea-level reefs are the central islands within oceanic barrier reefs, or the mainland coast within a continental barrier reef. The testimony of these witnesses has been too largely neglected, apparently because most investigators of coral reefs have been zoologists, little trained in the physiography of shore lines. Elevated reefs afford additional testimony in their structure and in the relation of their mass to its foundation; but these witnesses also have been insufficiently considered, perhaps because most investigators of reefs have, as zoologists, been little trained in structural geology; hence it seemed desirable to give as much time as possible on the Pacific islands to questioning the independent witnesses above designated, rather than to the study of the reef themselves.

The testimony of the first group of witnesses —the central islands of barrier reefs—convinced me that Darwin's theory of subsidence is the only theory competent to explain not only the development of barrier reefs from fringing reefs, but also the shore-line features of the central (volcanic) islands within such reefs; for the embayment of the central islands testify emphatically to subsidence, as Dana long ago pointed out: thus my results in the study of this old problem of the Pacific agree with those of several other recent students, especially Andrews, Hedley and Taylor of Australia, and Marshall of New Zealand. Darwin's theory of subsidence also gives by far the most probable explanation of atolls; for it is unreasonable to suppose that a subsidence of the ocean bottom should occur only in regions where the central islands of barrier reefs are present to attest it, and not in neighboring regions where reefs of identical appearance,