SCIENCE

FRIDAY, NOVEMBER 20, 1914

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THE MATHEMATICIAN IN MODERN PHYSICS[1]

IT is perhaps presumptuous for an experimental physicist to address a body of mathematicians. He can at best appeal. They are the arbiters of his science. They determine the number of cubic feet allotted for his antics. In a genial mood, they may give him the equivalent number of cubic centimeters. Physicists appreciate the clemency. Let nobody contend that there are, *necessarily,* laws of nature. In science, as in civil law, the experts in a measure make the facts. So I appeal to the law-givers of physics, with a purpose of exhibiting something of the method with which they have supposedly treated me, in the past forty years of my experience. If I am obtrusively personal I must be pardoned, for this is the only experience I have to give.

We, the experimentalists, are supposed to be the artists of science, a type of men who reach conclusions by intuition, by a happy leap in the dark. The inventor, the laboratory hermit, parades an essentially feminine type of mind, whereas the eternal masculine, the essentially logical trenchency, belongs to the mathematician. In all humility, however, in the dark recesses of the laboratory, there are skeptics who believe that both the physicist and the mathematician, in the main, follow the method of trial and error; that both develop from idea to idea. The usual outcome in the mathematical case is a huge paper basket, overflowing and standing in the waste; the outcome in the other, a sort of dismal morgue, a junk-shop of botches. Failures have been the rule, successes the exception. But as we flaunt our successes (and they only

[1] From an address given at the dinner of the American Mathematical Society, in Providence, September, 1914, on the occasion of the one hundred and fiftieth anniversary of Brown University, by Professor Carl Barus.

can be indefinitely manifolded, like truth) while our failures are still-born, we are known not for what we are or actually do, but by the occasional incident, by the happy accident. And it is this incident that wills it that the results of the mathematician are much more glorious, soaring unfettered and free even into transcendental space, whereas the results of the physicist, as a rule, must be of the earth, earthy. While the mathematician indulges an oriental dream—"Nein, Wir sind Dichter!" cries Kroneker—the physicist must tread the straight and narrow path, guided by the arithmetic of the fathers. We are the Puritans, you the unmitigated voluptuaries.

Judge, therefore, the astonishment of the world that it was left to our brother of the soil to detect the four-dimensional world among the inadequacies, or shall I say the débris, of the three-dimensional. It is the journeyman of science that clamors for a wider scope. It is rule of thumb evidence that cries exultingly "we are living in a Copernican era." Things believed to be at rest are asserted to be moving and so uncannily moving, that if there were an Inquisition in power to-day, we should all, like Galileo, be put to the torture. Need we then blame the physicist if in his intoxication he suspects that the mathematicians may not, after all, be the only experts, that the laws of his undoing may have been, in a measure, of his own making?

However, I am digressing too far. I will, therefore, reconsign the experimental skeptic to the allurements of his workshop and there he may grumble as he chooses.

I implied that it is suspected that the mathematician makes our laws for us. It is thus necessary to indicate, however superficially, what I mean. When I began my work in Germany in 1876 the theory of Weber, "Das electrodynamische Grundgesetz" of 1846, was rife in that country and had even invaded France (*cf.* Briot, Thermodynamique) and the other countries of continental Europe. Electrodynamics through the genius of Ampère (1821–22), had already definitely captured magnetism. Weber embraced the whole of electromagnetics in a single equation, consistent with the law of the conservation of energy. It was a beautiful theory; but it was action at a distance gone mad. Such indeed was the rage of these theories at the time that even Gauss and Riemann did not escape temptation, while Clausius revised and modified the argument throughout, bringing out a new theory of his own. I doubt whether any one here has read that theory. I have never seen it referred to and yet it is a superb piece of vigorous mathematical reasoning, quite worthy of Clausius.

I am induced to pause for a moment to speak of Weber himself, a singularly lovable child-like man, to all appearances hopelessly unpractical, so much so, that many of his intimates were wont to poke fun at him. But Weber, like his friend Gauss, was a profound mathematical thinker and in that capacity introduced two of the most practical things which the practical world has inherited; for the electric telegraph is a Gauss-Weber invention (1833); and what we now call our C.G.S. system of units is fundamentally the creation of Weber (1852) again following Gauss (1832). A man may, therefore, be practical even if he sometimes fails to drive a nail straight.

To resume: what these men did was to postulate a force which depended upon the states or motion of the point where force originates; but any phase of the force hammers away at any distant point co-temporaneously with the time of its origin. These electrical forces, in other words, did what gravitational forces still persist in doing. If we glance back at such theories from our present point of vantage, we can not but marvel how perilously near they came to the state of the case as we know it to-day. If they had only retarded their potentials! It is all the more curious that they suspected nothing, as the 3×10^{10} velocity which characterizes the relation of Weber's electrostatic to his electromagnetic system of units, measured by Weber and Kohlrausch in 1856, is the velocity of light.

The respite accorded to any of these theories was brief. In England they were vigorously condemned. Thomson and Tate, T and T', as we used to call them, in the earlier edition of their book anathematized them, as all the more pernicious in proportion as they were beautiful. They were completely swept away by the profound originality and incisiveness of the Faraday-Maxwell hypothesis (1854). Maxwell's great book (1873) had in fact appeared three years before I entered as a student, but it naturally was looked at askance in Germany, Helmholtz alone excepted. The aim of the earlier thinkers, to reduce the whole of electrical science to one equation was now to be realized in a way that marks one of the most important epochs in the history of physical science; an epoch comparable only to that of Newton; for although Maxwell modestly ascribes the incentive to his great accomplishments to Faraday, and believes that he is seeing nature with a mathematically unsophisticated eye, the capital discovery of the equations of the electromagnetic field (and this is the real issue) is Maxwell's creation. More than the widest sweep of the generalizing fancy could have anticipated was here completed; for at a single stroke of the wand, as it were, the whole domain of light and heat was annexed to electricity. It interpreted the meaning of the transparent and the opaque body of reflection and refraction. It introduced a new cosmical force, the light pressure long after found by Lebedew (1900), and our own countrymen, Nichols and Hull. It harmonized the divergent views of Fresnel and Neumann, admitting both impartially, and it gave to optics a new lease of life by lifting it over the obstructions of the elastic theory. Indeed Maxwell's best friends were apprehensive, since the theory predicted even more than was believed to exist, until in 1877 the new Maxwellian light dawned upon the mind of Hertz. The theory endowed the world medium, the ether, with new potencies, in insisting on its continuity, on the point to point transfer of electric force, so that ether stress became one of its familiar images, a veritable charm to conjure by.

It would carry us too far if we attempted to analyze the reaction of the new views on kindred sciences. Hydrodynamics, which had suggested the useful conception of the force-flux, in particular, profited and such beautiful researches as those of the Bjerknes (1863 *et seq.*) father and son, were stimulated in proportion as they fitted into the electromagnetic scheme. It was inevitable, moreover, that in the further treatment of Maxwell's equation the use of vector methods of computation should become indispensable in physics. They were approached cautiously enough and at first rather regarded as an affectation. Maxwell himself merely indicated the use of quaternian methods. Helmholtz, so far as I know, made no use of them. But in spite of petty differences of notation which still persist, the vector method became more and more general until to-day it is a commonplace, and beginning to make room for the new and more powerful 4, 6 and 9 dimensional geometry of higher vectors.

This was the second epoch and an epoch of unexampled fruitfulness. The ether electrically ignored heretofore has become all embracing. Woe to him that lisps, action at a distance! That Maxwell should have died before the ultimate vindication of his theory on the part of Hertz or the appearance of important corollary of Poynting (1884) is one of the tragedies of science. Similarly Hertz was not to witness the spectacular development of radio-telegraphy which followed so soon after his death. Maxwell's theory, which according to Hertz means Maxwell's equations, thus includes the whole of physics, dynamics alone excepted, and the world equation has advanced another step. Maxwell indeed, following the established custom, endeavored to call dynamics to his aid; but here his questions were put to a silent sphinx, inasmuch as mechanics had no counsel to give. Naturally the theory so revolutionary gained headway but slowly on the continent of Europe and even in England, unfortunately, Kelvin and (I believe) Rayleigh long remained unconvinced. When therefore the theory was universally accepted, it was already ripe for the modification, which Hertz himself actually began.

The ether as Maxwell left it has two independent properties, specific inductive capacity and permeability, which may be regarded as associated in the velocity of the electromagnetic wave passing through it. But the equations apply only for a medium at rest or at least approximately at rest, to a quasistationary medium. It is fortunate that a very coarse approximation to rest suffices; otherwise the early workers would have lacked encouragement. The new epoch, now about to dawn, thus found its point of departure in the motion of electrical systems. It has been in the main an era of confusion and bewilderment and one was to learn the hopelessness of any fundamental proof in physics. Instead of subjecting physics to the arbitrament of dynamics, we see dynamics pleading at the gates of electrical science, when electricity, distraught within itself, has no fundamental interpretations to offer. The troubles begin with the study of the first-order effects of moving optical systems, in the researches of Fizeau (1851); they become grave in the famous experiment of Michelson (1881) where the effects to be observed are of the second order. The speed of the earth, regarded optically from axes fixed in the ether, is zero. The ether and the earth have no relative velocity. This is tantamount to a rejection of the ether. Judge the consternation! As Maxwell's equation contained no direct reference to the motion of the charged body, a first attempt as I have already intimated was made by Hertz (1890) to supply this deficiency; but it was not of permanent value. The real interpretative advance came from Lorentz, in 1892. Although he fully realized and had endeavored to explain away the Michelson difficulties, Lorentz none the less boldly put his coordinates in an absolutely fixed ether, penetrating all bodies, even the atoms. He then went back to the methods of Weber, but with this essential difference that he included the whole dictum of the Maxwellian electro-magnetics in his postulates. The peculiar feature of the ether, its permittance and permeability, were abolished and in their place appears the velocity and density of the electron, or charged particle. Electric fluid exists; magnetic fluid does not. Lorentz then showed with consummate skill that the equations of the classic electromagnetics of Maxwell could be retained, that both the scaler potential and the vector-potential would retain their original form, would be invariant, so to speak, if the time-variable were belated by the interval consumed by light in passing from the source to the point of application in question. The profound originality and power of this and the earlier Lorentz transformation would perhaps not have been detected so soon, but for the unexampled abundance of new resources accruing to experimental physics at this time. In 1892 Lenard had isolated the cathode ray; Röntgen in 1895 discovered the X-ray. As a sort of corollary of the X-ray came the Becquerel-ray in 1896; the radium of the Curies in 1897, soon to be interpreted as to radiation by Thomson and Rutherford. The year 1896 brought the Zeeman effect, virtually predicted by Lorentz. The year 1898 brought Thomson's electron. In these and similar researches, bodies moving with a speed approximating that of light (easily exceeding $c/10$) were for the first time in history, at the disposal of the investigator. The new bodies, showing an inertia or virtual mass depending in a pronounced way on their speed, made havoc with Newton's laws and swept the classic dynamics mercilessly out of the field, as an arbiter of world phenomena. Theories such as those of Lorentz, 1892, or of Larmor, 1894, were now the only refuge. What could they do, was the ardent question, to replace dynamics?

Following the suggestion of Lorentz that the moving system contracts in the direction of motion, or at least apparently contracts to the fixed witness, Einstein in 1905 was the first to clearly perceive the iron logic of the situation; and the logic of a desperate situation is all there is in the theory of relativity. Einstein saw that if systems were to be interconsistent, time periods in the moving system would have to expand in the same second-order ratio to the ken of the fixed observer, so that time specifications and time frequencies may proportionately contract; or that identical clocks in the moving system must go slower. In such a case, any natural phenomenon, preferably a vacuum phenomenon like the velocity of light, is the same in all systems, moving or at rest. One system is as good as another. All observation is relative. The equations of this celebrated principle of relativity, culminating in Einstein's famous addition theorem of velocities belonging to different systems—an ultimate break with the Galileo transformation, where time has the same absolute value everywhere—have been the very focus of discussion for the last ten years.

In its original form, the principle is as yet rather a detached statement, adapted to definite purposes but lacking in mathematical elegance. It was left to the genius of Minkowski (1908) to mould this flotsam of ideas into a philosophical system of extraordinary symmetry and breadth, the promise of which it is, as yet, too soon to adequately appreciate. In fact, the untimely death of Minkowski was an irreparable loss to science, even if with Hilbert we resignedly conclude to be grateful for what he has done for us. Minkowski's world, as he himself remarks, is a response of modern mathematical culture to the urgent demands of the laboratory, and therein lies its strength. In the minds of prominent thinkers it is a philosophical revolution, an inversion of thought, as far-reaching in scope as the similar revolution of Copernicus. "Let space and time be submerged," cries Minkowski in an impassioned utterance, "Sie sollen in den Schatten versinken," to make way for a single unified world; in other words, let the incantation ring in a world in which the variables x, y, z, t, are linked with ties as inherent and indissoluble as the variables x, y, z, in common space. So understood, every point in space, even if at rest, describes a world line, which may be referred to and is contained between the two extremities of the time axis. Uniform motion is a straight world line. Any other motion an appropriately curved world line. World time is the length of a world line in relation to the speed of light. These world lines are thus a veritable warp and woof of the Deity. With Goethe we may say "Sie weben der Gotheit ewig Gewand"—or recall the curious passage of Wagner's Parsifal "Du siehst mein Sohn, zum Raum wird hier die Zeit."

To establish the connection between the four variables which shall be invariant in case of linear time transformations as is the case in Newton's dynamics, or that shall embrace the Einstein transformations as a special case, Minkowski postulates a four-dimensional hyperboloid with a single parameter c, the velocity of light, given by the reciprocal of the time axis. The other parameters are one. The hyperboloid is now usually made equilateral by calling the time variable ct. The intersection of the xt-plane with this hyperboloid, thus cuts out two hyperbolas symmetrically above and below the x-axis, the former (for positive time) alone being considered. The major axis is again the reciprocal of c, the minor axis a unit.

Now if the hyperbola in question with its parameter c is referred to conjugate diameters, it is easily shown that the oblique time and x-axes imply all the transformations of the theory of relativity, for the same c. The equation of the hyperbola is an invariant with relation to the new axes. The axes, or units of measurement, are proportionately increased, the specifications or numerics decreased, but the ties of the variables are exactly the same as before. Minkowski calls this the group G_c. Velocities greater than c are imaginary and are thus essentially excluded.

On the other hand, if the parameter c be supposed to increase to infinity, the symmetrical hyperbola eventually coincides with the x-axis, eliminating the time axis, and referring the whole system back to Newton's dynamics. This is the transitional group G^{∞}.

The generalized time is then the new variable of which x, y, z and t are all functions. Every translational vector now has four components and the rotational vectors six components, corresponding to the six pairs of variables or planes of rotation. One may even add that the new world, like Cæsar's Gaul, is divided into three parts by the asymptotic cones unknown to Cæsar. Axes may be so chosen as to make any two events contemporaneous. They need merely be parallel to the time axis selected. Similarly there are four equations of motion, the fourth being the energy equation, as energy itself is possessed of inertia. Finally, the equations of electromagnetic field in their magnetic and electric aspects, like the rotations, are given by the geometry of a vector with six components.

The treatment of motion is thus profoundly generalized, and Minkowski remarks that if these new transformations had been discovered by a mathematician "aus freier Phantasie," by an untrammeled imagination, they would have constituted a triumph in mathematics of the very first order. But, even under present circumstances, as soon as such developments were demanded by the laboratory, finding that within the atom the Newtonian world is certainly discredited, mathematics was at once ready to embody the new conception in a way that makes the bonds of mathematics and physics closer than before.

Vast and beautiful as these generalizations are, we must nevertheless confess that they are still but a coarse reproduction of nature; for in none of them is there any unequivocal or imperious demand for gravitation. Gravity still acts at a distance, as did the electrical vector in the days of Weber. Nor is the most generalized electromagnetic field able to account for the spectrum distribution of radiation, in the development of which energy threatens to pursue, if it has not already entered, the route of atomistic physics occupied by chemistry.

While mathematics is easily able to cope with the problems of relativity, even in their most generalized aspects, since they never break with continuity, the questions are more menacing in the second class of the recent demands of experimental physics, which came to a crisis in certain straightforward experiments on radiation made at the Reichsanstalt (Lummer and Pringshen, 1899; Christianson, 1884). The question dates back to Kirchoff's black body (1859), in which emission and absorption are equal. Some time after came Stefan's universal law of black body radiation (1879) and the theoretical verification on the part of Boltzmann in 1884. There was a period of intermission, in which the question of the equi-partition of the energy of a gas among the degrees of freedom of its molecules was vigorously discussed but without leading to available conclusions. However with the introduction of the black body by Kirchoff and the treatment of its radiation as a case of thermodynamic equilibrium, it was possible to assign both temperature and entropy to such radiation. But there was one further fundamental step to be taken and that was the definition of entropy apart from the Carnot engine and the intelligent manipulator, who is always an implied part of that wily machine. The second law was to be freed from reference to anything of a biological nature. Helmholtz had often insisted that the second law is the result of the order of physical size of the agent, in comparison with the atomic size, of his lack of equipment to control the individual molecule. To a being of molecular dimensions, there would be no irreversibility; whereas irreversibility has a very real meaning to the grosser attributes of the corrupter of nature. It was to the genius of Boltzmann (1877) that the fulfilment of this task was allotted. He was the first to give to entropy a purely mathematical signification, defining it as the logarithm of the probable occurrence of any thermo-dynamic state, be it a distribution of velocities, be it a definite distribution of discontinuous radiation energy-elements. Along this line, therefore, the new thermodynamics proceeded effectively. The first step came from W. Wien, whose displacement law of 1893 is embodied in the shift of the maximum of spectrum energy density, from red to violet, with increasing temperatures. Wien showed that a universal function of the ratio of temperature to frequency must here be in question. The determination of this universal function was the culmination of the insight and consistent labors of Planck (1900), who by postulating the energy quantum, became the creator of modern thermodynamics; for this energy element is a saucy reality, whose purpose is to stay. It not only tells us all we know of the distribution of energy in the black body spectrum in its thermal relations, but it gives us, indirectly, perhaps the most accurate data at hand of the number of molecules per normal cubic centimeter of the gas, of the mean translational energy of its molecules, of the molecular mass, of the Boltzmann entropy constant, even of the charge of the electron or electric atom itself. Under the guidance of Nernst it has created new chapters in the treatment of specific heats at low temperatures, their evanescence at the absolute zero of temperatures, the evanescence of the specific electrical resistance at zero, all more or less bearing on Dulong and Petit's law. Not less vital is the introduction of the new universal constant hitherto not even suspected, the "Wirkungs quantum," an equivalent of the Hamiltonian integral of action. Here then is a departure from continuity postulated for energy, which will hereafter operate with definite finite elements only. The condition of occurrence of such elements in any definite relations, can for this reason be specified as a case of probability.

Of the Planck molecular oscillators I must speak briefly. If operating continuously under the established electromagnetic laws they lead to the impossible distributions of energy in the spectrum investigated by Rayleigh and Jeans. But if emitting only, when their energy content is a whole number of energy elements, a case thus involving the entropy probability of Boltzmann, Wien's law and the numerical data referred to are deducible with astounding precision.

This then is the peculiar state of physics to-day. The appearance at the very footlights of the stage, of a new constant, the meaning of which nobody knows, but whose importance is incontestable. Moreover energy is seen there under an entirely new rôle. Grasping at greater freedom she has hopelessly involved herself in the meshes of the doctrine of probability. There was a time, the time antedating Mayer (1840–42) and Joule (1843), Kelvin and Clausius, when to speak of indestructible energy would have been rash. It was a glorious epoch when she first appeared in the full dignity of her conservative and infinite continuity. In contrast with this, the energy of the present day is scarcely recognizable. Not only has she possessed herself of inertia, but with ever stronger insistence she is usurping the atomic structure once believed to be among the very insignia of matter. Contemporaneously, matter itself, the massive, the indestructible, endowed by Lavoisier with a sort of physical immortality, recedes ever more into the background among the shades of velocity and acceleration.

But the single equation of nature, aimed at by Lagrange and Hamilton, by Weber and Maxwell in their several ways, has nevertheless throughout all this turmoil reached a more profound significance and now even holds dynamics, awkwardly it is true but none the less inexorably, in its grasp. That it is not complete, that it never can be complete, is admitted (for the absolute truth poured into the vessel of the human mind would probably dissolve it); but that it is immeasurably more complete to-day than it was yesterday is as incontrovertably true as it is inspiring.

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CONTEMPORARY UNIVERSITY PROBLEMS[1]

THE story of Clark University during the quarter century of its existence, the close of which we celebrate to-day with the alumni, under the inspiring guidance of Dr. French and his committee, has in some respects no parallel in academic history. Especially the first few years of our annals have both brighter and darker pages than I can find in the records of any university. Thirteen of us instructors had taught or taken degrees at the Johns Hopkins, and we left that institution, which had added a new and higher story to the American university, when it was at the very apex of its prosperity and hence were naturally

[1] Address given on the occasion of the celebration of the twenty-fifth anniversary of Clark University by Dr. G. Stanley Hall, president of the university.