SCIENCE

FRIDAY, JANUARY 23, 1914

CONTENTS



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## *THE INFLUENCE OF FOURIER'S SERIES UPON THE DEVELOPMENT OF MATHEMATICS*[1]

IN selecting a subject for to-day's address I have had the difficult task of interesting two distinct classes of men, the astronomer and the mathematician. I have therefore chosen a topic which, I trust, will appeal to both—trigonometric series. Though I propose to treat it only in its mathematical aspects, I shall try to do so in a broad way, tracing its *general* influence upon the trend of mathematical thought.

As you know, the theory of the infinite trigonometric series,

$$\text{(I.)}\quad f(x) = \tfrac{1}{2} a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

is different *ab initio* from that of the power series,

$$P(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

For the latter the fundamental element is $x^n$, of which the graph is, for positive $x$, a monotone increasing function, wholly regular, without peculiarities of any sort. It is therefore in no way surprising that the power series obtained by combining terms of form $c_n x^n$ define the most civilized members of mathematical society—the so-called analytic functions—which are most orderly in their behavior, being continuous throughout their "domains," possessing derivatives of all orders and a Taylor's series at every point; and so forth. On the other hand, the graph of $\sin nx$ or $\cos nx$ is a wave curve with crests and troughs, whose number in any $x$ interval increases indefi-

[1] Address of the vice-president of Section A—Mathematics and Astronomy, American Association for the Advancement of Science, Atlanta, 1913.

nitely with $n$. Accordingly, the functions defined by infinite trigonometric series are obtained by compounding waves of varying intensity and different wave-lengths and may be almost infinitely complicated in their behavior. This fact was fraught with vital consequences for mathematical development.

A further distinction between the trigonometric and power series appears in respect to the values which their argument may take. The convergent power series $P(x)$ has significance for at least a limited domain of imaginary values of $x$; on the other hand, it is possible for trigonometric series to define functions which have no meaning except for real values of $x$. As, therefore, the trigonometric series has a functional content totally different from that of the power series, its influence was felt first, and primarily, in the development of the notion of a function of a real variable.

The concept *function* was at first vague, as vague and indefinite as our geometrical intuitions. It had its root in the 17th century in the analytic geometry of Descartes. Here the variation of $y$ with $x$ along a curve inevitably suggests the notion of a function. The first published definition of the term appeared in 1718 when John Bernoulli defined a function of a variable as "*an expression which is formed in any manner from the variable and constants.*" Thirty years later, in his "Infinitesimal Analysis," Euler defined it in like manner except that the function is now an "*analytic expression.*" What is meant by "analytic expression" is not explained, but from his definition of special classes of functions it would appear that the term denoted an expression put together in terms of the variable and constants by a finite or infinite number of operations of addition, subtraction, mulitiplication, and division. Differentiation and integration were also undoubtedly permissible.

About this time there began the famous controversy over the mathematical representation of a vibrating string. This satisfies the well-known differential equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2},$$

where $a$ is a certain constant, $x$ the position of a particle on the string when taut, and $w$ its transverse displacement at time $t$. A solution of this problem for the case of fixed end points was given by d'Alembert in 1747 under the form

$$w = f(x + at) - f(at - x),$$

where $f(x)$ denotes an arbitrary function whose nature he apprehended too narrowly. But he claimed to have the general solution inasmuch as his solution involved an *arbitrary* function.

This shot into mathematics the question: *What is an arbitrary function?* Even to-day this question is a vexing one, owing to disagreement in the point-set theory concerning certain principles of logic which cluster around the "*Princip der Auswahl*" as a center. But mathematicians had not then arrived at the subtleties of the present day. Their difficulties were really caused more by imperfect notions concerning a function than by the degree of arbitrariness. On the basis of the above definition of a function then current, Euler maintained that d'Alembert's solution was particular, rather than the most general possible. He rightly apprehended the nature of the physical problem and saw that the motion of the string subsequent to the initial instant was completely determined by the initial form of the string and the initial velocities of its points. Now the initial shape of the string could be a continuous geometrical curve composed of successive pieces whose forms are absolutely independent of one another. To represent these pieces, Euler claimed that an equal number of different analytic expressions, or arbitrary functions, were necessary. Hence, as d'Alembert's solution involved only *one* arbitrary function, it could not be the general solution of the problem.

In these considerations of Euler there is a sharp antithesis between geometry and analysis. In Euler's thought the independent pieces of the above curve formed "*curvæ discontinuæ seu mixtæ seu irregulares.*" There was a blind belief that the definition of a curve in any interval by a mathematical expression carried with it a definite continuation of the curve beyond the interval, the violation of which was a violation of analysis. Thus the question was raised as to the relative power of mathematically constructible expressions and of geometric representation, and it was decided that geometric form transcends analytic expression rather than the converse.

The dual character of this controversy was changed into a triple one by Daniel Bernoulli, who first introduced Fourier's series into physics and obtained the solution of the equation of the vibrating string with fixed end points under the form of a trigonometric series,

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l},$$

where $l$ denotes the length of the string. The separate terms of this series give the tones and overtones of the vibrating string. Inasmuch as this solution is compounded of an infinite number of tones and overtones of all possible intensities, Daniel Bernoulli claimed that he had obtained the general solution of the problem.

For $t=0$ the above equation gives as the initial form of the string,

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}.$$

The question then at once arose whether d'Alembert's arbitrary function was capable of expansion into such a sine series. To Euler this seemed unthinkable. It was, so to speak, against the laws of the game, it was contrary to the rules of analysis that arbitrary, non-periodic functions could be represented in terms of periodic functions. Hence to Euler, Bernoulli's solution of the problem appeared even more limited than that of d'Alembert.

I have not the time to follow further this controversy, nor to show how d'Alembert and Lagrange united with Euler in declaring Daniel Bernoulli wrong in his claim. Yet not withstanding this overwhelming preponderance of authority Daniel Bernoulli was right. The controversy gradually languished without any clear conclusion till 1807, twenty-five years after Bernoulli's death, when Fourier presented to the French Academy one of the first of his communications which were summed up in 1822 in his "Analytic Theory of Heat." In this communication he startled Lagrange with the absolutely revolutionary doctrine that an arbitrarily given curve or function, irrespective of its nature, could be represented in any interval by a trigonometric series. Fourier sought no strict proof of his assertion, but the concrete examples which he gave vindicated its force. The precise limitations necessary to make the assertion exactly true remained, and to some extent still remain, for his successors to ascertain.

Fourier's result not merely vindicated Daniel Bernoulli's claim for his series, but showed that his claim fell far short of the reality. At a single blow it shattered hopelessly the notion of Euler and his contemporaries that a mathematical function could be carried continuously beyond the interval of definition in only a single way. But Fourier's examples went further than this. The arbitrary curve which he represented by his series (I) could consist of separate pieces of any sort, not merely having no logical or definitional dependence on one another, but even not connecting successively at their ends. Thus by virtue of Fourier's assertion the power of representation through analytic expression is at least as great as the power of geometric picturization.

When once it was realized that mathematical expression could be adapted to the most diverse and unrelated demands upon it, no logical stopping-point could be seen short of the definition to-day accepted for a function of a real variable, and often referred to as the Dirichlet definition of a function. If, namely, to every value of $x$ in an interval there corresponds a definite value of $y$ (no matter how fixed or determined), $y$ is called a function of $x$. For example, $y$ may be equal to $+1$ at all rational points which are everywhere dense in any interval, and equal to 0 at the irrational points which are likewise everywhere dense. The Fourier series has thus necessitated a radical reconstruction of the notion of a function. *This is the first of its services which I wish to emphasize, the development and complete clarification of the concept of a function.*

Without loss of generality the interval in which the representation of the function by the series is required may be supposed to lie between $-\pi$ and $+\pi$. The series has then the form (I.) hitherto assumed. To determine its coefficients from the function Fourier used for the most part the equations,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx;$$

but this determination, as Fourier himself stated, had been made by Euler before him. Trigonometric series whose coefficients can be obtained from the function represented in this manner are now called *Fourier's series* in distinction from trigonometric series whose coefficients can not be so obtained through integration. I have, however, in the title of my paper used the term "Fourier's series" in the older and broader sense as synonymous with all series of the form (I.).

The consideration of trigonometric series from a strict mathematical standpoint marks a second epoch in their history. This began with Dirichlet in 1829 in a memoir remarkable for its combination of clearness and rigor. Here he first determined accurately a set of sufficient conditions for the expansion of a function into a Fourier series. These familiar "Dirichlet conditions" it is scarcely necessary to repeat.

The extension of his results was at once sought, in particular by Riemann in a Göttingen Habilitations-Dissertation, which bore the title "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe." Riemann's aim was, however, to determine the *necessary* conditions for the representability of the function by the series. Must the function be integrable, as required in the sufficient conditions of Dirichlet? Must it have only a finite number of maxima and minima and of discontinuities? Such questions as these were easily answered by him in the negative, and a flood of light was poured upon the problem of representability but without making visible its complete solution. Possibly it was for this reason that this Habilitationsschrift, though delivered in 1854, was not published until thirteen years later, and then only after Riemann's death. Yet the work is a classic. As has been said of the poet Coleridge, so it could be said of Riemann, he wrote but little, but that little should be bound in gold.

To put the theory of Fourier's series on a broader basis, Riemann perceived that first of all it was necessary to sharpen and widen the concept of an integral. Initially Leibnitz had thought of integration as a summation process, but this notion was forced into the background by its definition as the reverse of differentiation, until revived by Cauchy in 1823. He then defined the integral of a *continuous* function as most of us were taught to define it. The interval of integration was divided into $n$ parts $\delta_i$, each $\delta_i$ was multiplied by the value of the function $f(x_i)$ at its beginning, and the integral was defined as the limit of the sum $\Sigma\delta_i f(x_i)$ when the number of parts increased indefinitely, their size diminishing indefinitely. Because of the continuity of the function this definition of the definite integral was equivalent to that framed by means of the reverse process of differentiation. Riemann dismisses altogether the requirement of continuity for the function, and in forming the sum multiplies each subinterval $\delta_i$ by the value of the function, not necessarily at the beginning of the interval, but at a point $\xi_i$ *arbitrarily* assumed in the subinterval. If, then, a limit exists for the sum $\Sigma\delta_i f(\xi_i)$, irrespective of the manner of partitioning the interval and of the choice of the points $\xi_i$, this is called the integral. Thus he redefined the fundamental concept of the integral calculus, making it entirely independent of the differential calculus. This definition, often referred to as the Riemann definition of an integral, has now become the universally accepted one and is the basis of scientific treatment of the integral calculus. *Thus a second service of Fourier's series has been in laying the foundation of the modern integral calculus, and in such wise that it bid fair to completely eclipse the differential calculus in importance and reach.*

Riemann's memoir may also be characterized as the beginning of a theory of the mathematically discontinuous. The work of Fourier had disclosed that mathematical expressions could portray functions with breaks, and the exacter but more limited investigation of Dirichlet drew still further attention to discontinuities. Riemann's definition of an integral did more; with one leap it planted the discontinuous function firmly upon the mathematical arena. In his integrable functions was comprised a class of functions whose discontinuities were infinitely dense in every interval, no matter how small—though indeed, as we now know, they are not totally discontinuous. One example which he gave was the integrable function defined by the convergent series,

$$1 + \frac{(x)}{1^2} + \frac{(2x)}{2^2} + \frac{(3x)}{3^2} + \cdots,$$

in which $(nx)$ denotes the positive or negative difference between $nx$ and the nearest integral value, unless $nx$ falls half way between two consecutive integers, when the value of $(nx)$ is to be set equal to 0. The sum of the series was shown to be discontinuous for every rational value of $x$ of the form $p/2n$, where $p$ is an odd integer relatively prime to $n$.

This example and others, such as that of an integrable function with an infinite number of maxima and minima which was incapable of representation by a Fourier's series, were exceedingly stimulating. The investigation so impressed the imagination of Hermann Hankel as to call forth his notable memoir "Über die unendlich oft oszillierenden und unstetigen Functionen" in which he unfolds his principle of "condensation of singularities," a memoir so important that it has even been said to "entitle him to be called the founder of the independent theory of functions of a real variable." It would appear to me that this distinction could be assigned with equal propriety to Riemann, for historically the first of the two or three principal sources of this theory is to be found in Riemann's application of integration to discontinuous functions in his memoir on "the representability of a function by Fourier's series."

The above example of Riemann is notable for giving a mathematical expression for a discontinuous function incapable of graphical representation. I have already pointed out how Euler conceived of graphs so arbitrary as to be impossible of representation through an "analytic expression." The scales were now turned decisively to the other side, though it was not till later that it was recognized that our geometric figures have only an approximating character which our mathematical equations refine.

But the full power of mathematical expression was not realized until 1872–1875, when Weierstrass startled the mathematical world with an example (first published by Du Bois Reymond in 1875) of a continuous function having nowhere a derivative, or, in other terms, of a continuous curve without a tangent. The function given by Weierstrass was a trigonometric series

$$\sum_{n=1}^{\infty} b^n \cos a^n \pi x,$$

in which $b$ is a positive constant less than 1 and $a$ a fixed odd integer large enough to make $ab$ exceed a certain value. Weierstrass states also that Riemann is supposed to have shown that the series

$$\Sigma \frac{\sin n^2 x}{n^2}$$

represented a function of like property, but the proof was not known. The failure of the continuous function of Weierstrass to be differentiable is due to the possession of an infinite number of maxima and minima in any interval, however small.

This example completed the separation between differentiable and continuous functions. It shows that *the former are only a subclass of the latter,* a result not even surmised by the boldest geometrical intuition. *This is the third influence of Fourier's series which I wish to emphasize.* So far as I know, this is the only one of its results which vitally affects geometric theory. It reveals the transcendence of analysis over geometrical perception. It signalizes the flight of human intellect beyond the bounds of the senses.

I return now to trace further the march of the function theory of the real variable. The second principal element in its formation seems to me to have been the concept of *uniform convergence.* This also seems to have been suggested chiefly by study of trigonometric series. Originally it was supposed that the sum of a convergent series of continuous functions shared the common properties of its terms and accordingly was continuous. Even so great a mathematician as Cauchy fell for a time into this error. The fallaciousness of this assumption was first pointed out by Abel in 1826 in his well-known memoir on the binomial series. Here he also discusses the series

$$f(\phi) = \sin \phi - \frac{\sin 2\phi}{2} + \frac{\sin 3\phi}{3} - \cdots, \qquad (2)$$

every term of which is continuous. Clearly the sum vanishes whenever $\phi$ is a multiple of $\pi$. If $\phi$ lies between $m\pi$ and $(m+1)\pi$, the sum is $\phi/2 - \nu\pi$, where $\nu$ denotes the

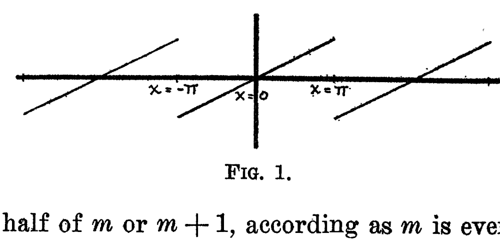


Fig. 1.

half of $m$ or $m+1$, according as $m$ is even or odd. Consequently, when $\phi$ passes through an odd multiple of $\pi$, the sum has a discontinuity of amount $\pi$, as is indicated in the adjoining graph. This result is in sharp contrast with the continuity of the sum which he demonstrates for the binomial and other real power series. At the same time he establishes the circular form of the region of convergence of the binomial series. Here, then, appears the initial cleavage between the theories of real and of analytic functions.

The difference between the trigonometric and the power series in respect to continuity is naturally to be sought in the character of the convergence at the points of continuity and of discontinuity. This difference was pointed out by Stokes in 1847 and by Seidel a year later. Both discovered the infinitely increasing slowness of convergence of the series on approaching a discontinuity of its sum. Consequently a discontinuity can not be enclosed in any interval, however small, in which the convergence is throughout "von gleichem Grade." In more modern parlance, the convergence is non-uniform. Seidel in his introduction explicitly points out that the erroneousness of Cauchy's conclusion (see above) is obvious from the existence of discontinuities in functions represented by Fourier's series, and he is evidently incited thereby to seek a cause for the discontinuity. The origin of Stokes's study is sufficiently obvious from its title: "On the Critical Values of the Sum of Periodic Series." His failure to appreciate the importance of his own convergence discussion is evident from the fact that it is not even mentioned in the opening analysis of his lengthy memoir.

A third discoverer of uniform convergence was Weierstrass, who is known to have been in possession of the notion as early as 1841. Through his followers (Heine and others) it gradually percolated into the mathematical literature. Unlike Seidel and Stokes, he thoroughly realized its importance. As Osgood has well said in his *Functionentheorie,* he developed uniform convergence into one of the most important organs "(methods) of modern analysis." The origin of the notion in the case of Weierstrass I have been unable to ascertain. A conjecture or surmise may therefore be pardoned. As is well known, the work of Weierstrass is rooted in that of Abel, the central theme or core being the theory of Abelian functions. It would not seem to be altogether improbable that both Weierstrass's theory of the analytic function and his concept of uniform convergence had as their starting point Abel's memoir on the binomial series. For here, on the one hand, with the demonstration of the circular form of the region of convergence of the binomial series, we find a proof of the continuity of the series which involves implicitly the idea of uniform convergence; on the other hand, we have in the footnote a series with discontinuities due, in fact, to non-uniform convergence. It would be a small matter for the discriminating Weierstrass to see that the continuity of the sum could not be carried over from the binomial to the trigonometric series, because there was not the same kind of convergence in the latter case. If this surmise is correct, the discovery of uniform convergence in the case of the third discoverer also is closely connected with a Fourier series.

I have dwelt at some length on uniform convergence because its discovery marks both the culmination of the first and older epoch in the treatment of functional series, and the beginning of a new one. In uniform convergence and a study of the discontinuous we have sought for the chief springs of the modern *theory of functions of a real variable. By so doing we are led to assign as a fourth great service of Fourier's series the genesis of this theory.* It is not to be forgotten, however, that other sources have also copiously contributed. The morphology of one member of a body must be in many ways perverted, if studied without correlation to the other members. But, after all, it is the Fourier series which gave the initial push and chief impetus to the construction of the function theory of the real variable.

This becomes still plainer if we take into consideration the comparatively recent point-set theory. Originally an off-shoot of the real function theory and still often treated by itself, it has been largely absorbed back into this theory, and its concepts already permeate analysis. Its founder was George Cantor, who was trained in the exact yet fertile school of Weierstrass. His earliest papers presaging this theory relate to a trigonometric series.

Two problems occupy his attention. The first is to show that if the series $\Sigma\ (a_n \sin nx + b_n \cos nx)$ is convergent throughout an entire interval, except possibly for a finite number of points, the coefficients $a_n$ and $b_n$ have for $n = \infty$ the limit 0. The second is to establish the uniqueness of the development of a function into a trigonometric series; in other words, to prove that when $\Sigma\ (a_n \sin nx + b_n \cos nx)$ is identically 0 over an interval, then each and every $a_n$ and $b_n$ must be 0. The requirement of convergence of the sum in the one case and of its vanishing in the other, was originally made for the entire interval, but Cantor found that it could be remitted for certain infinite aggregates of points without affecting the truth of the conclusions. He was led consequently to introduce the notion of the "*derivative of a point-set.*" Consider with him the set of points for which the requirement is omitted, and suppose that they cluster in infinite number in the vicinity of any point. This will be called a limit-point of the set. The totality of these limit-points is called the first derived set, or first derivative. This derived set of points may also have cluster points which form the second derivative; and so on. After introducing this concept, Cantor proved that the requirement could be remitted for any set of points whose $n$th derivative contains only a finite number of points and whose $(n+1)$th derivative accordingly vanishes.

In these very early papers of Cantor we have very clearly the beginning of his point-set theory. His attention is here concentrated upon an infinite aggregate of points, and the notion of the derived point-set was the first of the concepts by means of which he is able to distinguish between different infinite aggregates of points. Prior to Cantor no effort was made to distinguish qualitatively between them. To be sure, mathematicians were thoroughly conversant with the distinction between a continuous curve or set of points, on the one hand, and a merely dense aggregate of points such as the totality of points with rational coordinates. The raw material lay at hand for a beginning, especially in the work of Riemann and others on integration. Cantor alone saw the imperativeness of the need. In comparing infinite sets of objects and seeking a theory of the truly infinite he blazed a new path for the human mind. *As a fifth and a mighty influence of Fourier's series we have, therefore, to record the historic origin of the theory of infinite aggregates.*

Thus far in my sketch I have traced one strong, single current of influence of the Fourier's series. I have now to indicate some other effects without close relation to the foregoing.

In Fourier's "Analytical Theory of Heat" there are found what are said to be the first instances of the solution of an infinite number of linear equations with an infinite number of unknowns. He has, for example, to determine the coefficients in the equation:

$$1 = a \cos y + b \cos 3y + c \cos 5y + \cdots.$$

For this purpose he differentiates an even number of times, obtaining thus the system of equations

$$0 = a \cos y + b3^n \cos 3y + c.5^n \cos 5y + \cdots \quad (n = 2, 4, \ldots)$$

Combining this with the preceding equation and putting $y = 0$, he obtains an infinite number of equations of first degree with an infinite number of unknowns, $a$, $b$, $c$, $\cdots$. To solve these he uses the first $m$ equations to determine the first $m$ unknowns, suppressing all the other unknowns, and finally determines their limiting values as $m$ increases indefinitely. There is no time to point out the lack of rigor. Fourier uses his mathematics with the delightful freedom and naïveté of the physicist or astronomer who trusts in a mathematical providence.

This suggestive line of attack was not followed up, and indeed could not be, prior to the development of a theory of infinite determinants. When such a system of linear equations with an infinite number of unknowns came again to the foreground, the inciting cause was again a trigonometric series. I refer, as you know, to the work of our own astronomer, Hill. In his memoir on the "Motion of the Lunar Perigee" he had before him a differential equation of the following form, with numerical coefficients:

$$\frac{d^2w}{d\tau^2} = 2w\left(_2 + \theta_1 \cos 2\tau + \theta_2 \cos 4\tau + \cdots\right).$$

Assuming a solution in the form

$$w = e^{ic\tau} \sum_{n=-\infty}^{n=+\infty} b_n e^{2in\tau}$$

(which except for the factor $e^{ic\tau}$ is only a trigonometric series under another guise), Hill obtains for the determination of $c$ and the $b_n$ an infinite system of equations linear in the $b_n$. The elimination of the $b_n$ then gives $c$ as the root of a certain infinite determinant, and then the values of the $b_n$ are also found by use of infinite determinants.

The importance of Hill's results at once attracted the genius of Poincaré whose attention had, in fact, been previously drawn by Appell to an infinite system of linear equations. Poincaré now proceeded to consider the question of the convergence of infinite determinants, and in so doing laid a sound foundation for a new mathematical subject. In this new theory of infinite determinants the central thought is the passage, under restrictions to be properly ascertained, from a finite to an infinite system of linear equations. This principle here employed has been since applied in an even more striking manner by Fredholm, who was led through its use to his historic solution of a class of integral equations. In the theory of these equations the infinite determinant plays an indispensable rôle. *A sixth influence of Fourier's series is thus seen in the origin of a theory of infinite determinants,* also indirectly in the theory of integral equations for which it has supplied an important tool.

The seventh and the last influence on which I shall specifically dwell is more subtile, not so easily pointed out or demonstrated as some of the foregoing, but nevertheless one of the most far-reaching and probably the most pervasive of all. The physicist, astronomer, or mathematician has again and again to expand an arbitrary or assigned function into a series of functions, the nature of which varies with the problem before one. When once the idea and method of expressing an arbitrary function in series of sines and cosines have been won, they can be extended to other series of functions, as for instance series of Bessel's functions, zonal harmonics, Lamé polynomials, spherical harmonics. For such developments the trigonometric series with its applications has repeatedly served as a guide post. Numberless analogous results have been suggested thereby, though without definite statement of the fact. To take an example at random, the relation

$$\int_0^1 P_{2n}(x)P_{2m}(x)dx = 0 \qquad (m \neq n)$$

has its trigonometric analogues

$$\int_0^\pi \cos n\theta \cos m\theta \, d\theta = 0.$$

Who can deny, or who can affirm, in many such individual instances that the suggestion came from the trigonometric series? Yet in the bulk the debt is so great that he who runs can read it.

It is especially in connection with boundary value problems that we encounter series of functions. Now the trigonometric series was the inevitable tool for the first boundary value problems—those of vibrating strings, rods, columns of air, etc. Later, when Fourier crystallized the boundary value problems into classic shape, he used trigonometric series and, to lesser degree, similar series of Bessel's functions, obviously because these afforded him the simplest tools for the simplest problems. From series of sines of multiple angles he was led by certain problems in heat conduction to series of form $\Sigma c_i \sin a_i x$, where the $a_i$ are roots of a certain transcendental equation. Thence the orientizing influence of Fourier's series is continued down to the modern development of normal functions in the theory of integral equations. All such influences are in the very warp and woof of mathematical development and can not be disentangled. To minimize or ignore them would be to give a distorted picture. They form a most vital and leading part of the mighty theory of harmonic and normal functions and of the boundary value theory.

The extent of these influences in the past gives rise naturally to the question of whether the trigonometric series will continue to exert such a moulding influence in the future. Certain results of Baire to be shortly mentioned incline one to answer negatively. Yet the questions regarding the convergence of the series and the character of the functions which it can represent are even to-day incompletely answered. When new implements are invented, it is still to these unanswered questions that the investigator naturally turns to test their worth, as, for example, Lebesgue with his great new concept of an integral which has application when Riemann's integral is void of sense, or Fejér with a method of summing a divergent series. Also the Fourier series still offers an occasional surprise. Who indeed would have anticipated Gibbs's discovery, since extended by Bôcher, which relates to the approximation curve $y = S_n(x)$, obtained by equating $y$ to the sum of the first $n$ terms of the series (2) above? As $n$ increases indefinitely, the amount of the oscillation of the curve in the vicinity of each point of discontinuity of the limit does not tend toward the measure of the discontinuity, as would be supposed, but to this value increased in a certain definite ratio! But it may be reasonably expected that these surprises will become fewer and less important.

In this brief review I have neglected certain less analytic aspects, such as trigonometric interpolation and the use of the series in computation and in the perturbation theory. It has also not been necessary to emphasize the simplicity of structure of the series and its adaptation to computation. Neither do I need to speak of its correspondence in structure to so many periodic phenomena of nature, sound, light, the tides, etc. But I do wish, in closing, to emphasize and examine further, one aspect implied in all my preceding considerations, the wonderful *pliability* of the series.

It was this pliability which was embodied in Fourier's intuition, commonly but falsely called a theorem, according to which the trigonometric series (I.) "*can express any function whatever between definite values of the variable.*" This familiar statement of Fourier's "theorem," taken from Thompson and Tait's "Natural Philosophy," is much too broad a one, but even with the limitations which must to-day be imposed upon the conclusion, its importance can still be most fittingly described as follows in their own words: The theorem "*is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance.*"

Truly, the theorem is so comprehensive in its mathematical content that we mathematicians may well query with one of my colleagues whether it may not have conditioned the form of physical thought itself—whether it has not actually forced the physicist often to think of complicated physical phenomena as made up of oscillatory or harmonic components, when they are not inherently so composed.

It is this same pliability of the series that has been a source of perpetual delight and surprise to the mathematician. It has revealed an undreamt-of power in analysis. It has stimulated intuition and vigor, and has helped to usher in a modern critical era in mathematics similar in spirit to the Greek period. It has separated differentiable from continuous functions; it has put the integral calculus on a basis of independence of the differential calculus; it has focused attention upon sets of irregularities and discontinuities whose study has started the point-set theory; it has opened the field of discontinuous functions to analysis and, above all, has engendered a theory of functions of the real variable.

To the mathematician the theory of analytic functions for some time appeared to be of much greater importance than the freaky theory of the real variable, because almost all the important functions of mathematics are analytic. Also, the same has been hastily assumed for physics because the real and imaginary components of an analytic function are harmonic functions satisfying Laplace's equation. But this is to ignore features of at least equal, if not of superior, importance. Not long ago many thought that the mathematical world was created out of analytic functions. It was the Fourier series which disclosed a *terra incognita* in a second hemisphere.

Here, in the new hemisphere, the mathematician has advanced beyond the boundary of the trigonometric series. It has been found that discontinuous functions representable through such series form a thoroughly restricted class. They belong to what Baire calls the first class of functions which are limits of convergent sequences or series of continuous functions, themselves of "class 0." These in turn may be used to generate new functions. Even as non-uniformly convergent Fourier series may give rise to discontinuous functions of Class 1, so non-uniformly convergent series of functions of this class may give a new sort of functions of Class 2, and so on. Indeed, to every transfinite number $\alpha$ of the first or second class there corresponds, as Lebesgue has shown, a definite class of functions. *Thus the Fourier series has, after all, a very limited range of representation* *in the totality of functions mathematically conceivable.*

Even for functions of Class 0 or 1 the trigonometric series has a limited power of representation. This is manifest from an example given by Paul Du Bois Reymond of a continuous function which can not be represented by a trigonometric series. It remains to determine in the future just what properties are necessary and sufficient to characterize those functions of Classes 0 and 1 which are expressible by means of trigonometric series.

Earlier in my paper I pointed out that the generality of functions representable through Fourier's series was so great that the mathematician was led irresistibly to the Dirichlet definition of a function. If, namely, to every value of $x$ in an interval we have a corresponding value of $y$, then $y$ is called a function of $x$, no matter how the correspondence is set up, whether by a graph, a mathematical expression, a law, or any other way. To-day the pendulum has swung back to the old question of Euler. The study of representability in terms of trigonometric series has been succeeded by the broader question of the possibility of analytic expression in general. Now every continuous function, as is well known, can be represented by a uniformly convergent set of polynomials. Starting then from the totality of polynomials as a basis of functions for Class 0, we arrive successively at Baire's and Lebesgue's classes of functions corresponding to or, if you prefer, marked, by the transfinite numbers of the first and second classes.

Do these different classes of functions comprise all which are "*analytically expressible*"? Before answering the question it is necessary first to sharply define the phrase "analytically expressible." This is done by Lebesgue. Then, after broadening the content of these classes in a manner I have not the time to describe, he goes on to show that they do in truth comprise all such functions. The final question then confronts us: Are all possible functions included which are defined in accordance with the general definition of Dirichlet? In other words, are there functions *incapable of being "analytically expressed"*? Lebesgue by an example shows that this is the case. Our study of the Fourier series opened with the question: What is an arbitrary function? Here, at last, apparently, we have discovered the existence of a function of such a height or depth of arbitrariness as to be mathematically inexpressible. Having started with the Fourier series on a voyage of exploration, shall we conclude by saying that there is for us an unattainable pole?

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## UNIVERSITY REGISTRATION STATISTICS

THE registration returns for November 1, 1913, of thirty of the leading universities of the country will be found tabulated on the following page. Specific attention should be called once again to the fact that these universities are neither the thirty largest universities in the country, nor necessarily the leading institutions. The only universities which show a decrease in the grand total attendance (including the summer sessions) are Harvard, Western Reserve and Yale, the attendance of the two institutions last named having remained practically stationary. The largest gains in terms of student units, including the summer attendance, but making due allowance by deduction for the summer session students who returned for instruction in the fall, were registered by New York University (965), Illinois (944), Columbia (927), Wisconsin (749), Pennsylvania (681), California (614), Iowa (598),