DISCUSSION AND CORRESPONDENCE

THE PYTHAGOREAN THEOREM

TO THE EDITOR OF SCIENCE: In your journal for December 16, 1910, Dr. Northrup asks whether a dynamical investigation which he there gives is a proof of the Pythagorean theorem; and in the number for March 24, the question is discussed by Professor Deimel and Mr. Hersey. Looking at the question from the point of view of vector-analysis, or rather of the algebra of space, I would answer, Yes. Dr. Northrup starting from the principle of kinetic energy and certain other principles of dynamics deduces two expressions for the kinetic energy of the system shown in his diagram; and from the equivalence of these expressions he deduces the forty-seventh proposition of the first book of Euclid, commonly called the Pythagorean theorem; but he could with ease deduce the more general proposition (Euclid II., 12 and 13) which expresses the side of any plane triangle in terms of the other two sides and their included angle. His proof is merely the reverse of the following reasoning. I look upon the x, y, R, r and $-r$ of his diagram as vectors. The kinetic energy of the first mass is $\frac{1}{2}m(xW)^2 = \frac{1}{2}mW^2x^2$; and similarly that of the second mass is $\frac{1}{2}mW^2y^2$. But

$$x^2 = R^2 + r^2 + 2 \cos Rr$$

and

$$y^2 = R^2 + (-r)^2 - 2 \cos Rr$$

where $\cos Rr$ denotes the rectangle formed by R and the projection of r along R. Hence

$$\begin{aligned} \tfrac{1}{2}mW^2(x^2+y^2) &= \tfrac{1}{2}\, 2m(R^2+r^2)W^2, \\ &= \tfrac{1}{2}\, 2mR^2W^2 + \tfrac{1}{2}\, 2mr^2W^2 \end{aligned}$$

Here we pass from the one to the other expression for the kinetic energy of the system by means of the extended Pythagorean theorem; on the other hand, Dr. Northrup can deduce from the two expressions for the kinetic energy of the system the truth of this geometrical theorem.

This same principle that $E = \frac{1}{2}mv^2$ has an important bearing on the fundamental principles of vector-analysis: it places the orthodox quaternionist in a corner from which there is no escape. Because E is assumed in mathematical analysis to be positive and $\frac{1}{2}m$ is positive, it follows from the established principles of analysis that v^2 must be positive; consequently, to hold that the square of a simple vector is negative is to contradict the established conventions of mathematical analysis. The quaternionist tries to get out by saying that after all v is not a velocity having direction, but merely a speed. To this I reply that

$$E = \cos \int mvdv = \tfrac{1}{2}mv^2,$$

and that in these expressions v and dv are both vectors having directions which in general are different.

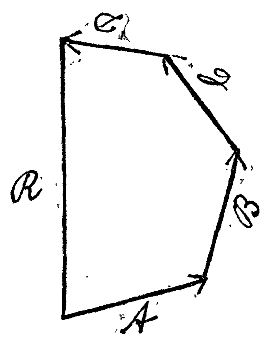

FIG. 1.

Recently (in the *Bulletin* of the Quaternion Association) I have been considering what may be called the generalization of the Pythagorean theorem. Let A, B, C, D, etc. (Fig. 1), denote successive vectors having any directions in space, and let R denote the vector from the origin of A to the terminal of the last vector; then the generalization of the Pythagorean theorem is

$$\begin{aligned} R^2 = A^2 + B^2 + C^2 + D^2 + & \\ + 2\{\cos AB + \cos AC + \cos AD + \} & \\ + 2\{\cos BC + \cos BD + \} & \\ + 2\{\cos CD + \} & \\ + \text{etc.} & \end{aligned}$$

where $\cos AB$ denotes the rectangle formed by A and the projection of B parallel to A. The theorem of Pythagoras is limited to two vectors A and B which are at right angles to one another, giving $R^2 = A^2 + B^2$. The extension given in Euclid removes the condition of perpendicularity, giving $R^2 = A^2 + B^2 + 2\cos AB$. Space geometry gives $R^2 = A^2 + B^2 + C^2$ when A, B, C are orthogonal, and $R^2 = A^2 + B^2 + C^2 + 2\cos AB + 2\cos AC + 2\cos BC$ when that condition is removed.

Further, space-algebra gives a complementary theorem, never dreamt of by either Pythagoras or Euclid. Let V denote in magnitude and direction the resultant of the directed areas enclosed between the broken line $A + B + C + D$ and the resultant line R, and let $\sin AB$ denote in direction and magnitude the area enclosed between A and the projection of B which is perpendicular to A; then the complementary theorem is

$$\begin{aligned} 4V = 2\{\sin AB + \sin AC + \sin AD +\} \\ + 2\{\sin BC + \sin BD +\} \\ + 2\{\sin CD +\} \\ + \text{etc.} \end{aligned}$$

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A BRIGHT AURORA OF SEPTEMBER, 1908[1]

THE finest display of the aurora borealis seen by the writer in Omaha during the last twelve years, took place on the night of Monday, September 28, 1908. Before describing its appearance, it may be of interest to mention the weather conditions that preceded and accompanied it.

No rain had fallen for about five weeks, the temperature during the day time had been unusually high, and strong winds had filled the air with disagreeable clouds of dust. The long duration of this state of the weather had become very monotonous. The expected clashing of a low from the northwest and of a West Indian storm from the southeast, had failed to bring any relief. Another week passed, and after an occasional cloudiness of the sky and an increasing humidity of the air had only tantalized this section of the country with unredeemed promises of moisture, the rain came at last and with it a rapid and great reduction of temperature, the thermometer fell from the eighties in which it had been hovering down to ten degrees above the freezing point. Monday, the day of the auroral display, opened with a temperature of about 40 degrees and a cold and disagreeable rain. It was still cloudy and misty at noon, but at about three o'clock the sky cleared, the wind came from the northwest, and the night began with a quiet, cloudless, cold and most transparent sky. An occasional glance at the heavens could detect no indications of an aurora. It was noticed rather suddenly at about 9:50 P.M. It then appeared as an arch extending from the northwest to the northeast horizon, and was about 8 degrees high on the meridian. Below the arch was a well-defined black space of uniform tint, which might easily have been taken for a bank of clouds. The arch itself was of a beautiful, soft, silvery whiteness, and seemed to be about 5 degrees in width. Its upper limit was not quite as distinct as its lower one. At this time there were no streamers of any kind, nothing but the arch. There was no moon to interfere with the display as it was seen from the observatory, and the city lights were also far enough away not to blind the eyes of the observer.

In about a quarter of an hour the scene changed. A few detached streamers now began to make their appearance, like the softened beams of search lights below the horizon. They were from about two or three to 20 or 30 degrees in length, and from one fourth to about 4 degrees in width. The short beams seemed to come directly out of the ground and were visible against or through the black space below the arch, and the longer ones passed visibly even through the bright arch itself. They did not seem to have any perceptible lateral motion, but they all seemed to come from the same vanishing point, which was estimated to be about 50 degrees below the horizon and on the meridian. The largest and broadest streamer was in the northwest, at the very end of the arch. It was about four degrees wide and 20 degrees long, and of a decided blood-red tint. A few of the other

[1] This excellent description was originally sent to the *Monthly Weather Review*, but is transferred to SCIENCE, as it belongs to cosmical physics rather than to climatology, to which the *Monthly Weather Review* is now confined.—Cleveland Abbe.