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CONTENTS

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THE INFLUENCE OF ASTRONOMY ON MATHEMATICS

THERE are probably many reasons why the members of the eleven sections of the American Association, representing at least fifteen sciences, have united in a single society. One of these is undoubtedly that the mingling of men of varied chief interests, points of view and methods of work has at least a tendency to correct those intellectual provincialisms which are characteristic of isolation, and to show how wide and how rich is the field of scientific activity. While it is unquestionably advantageous on some occasions for narrower groups of men whose interests are more nearly common and whose scientific activities run more nearly in the same channel, to meet apart for the consideration of their own special problems, yet on the whole the benefits to be derived from occasional joint meetings are so great that it is earnestly hoped the American Association will prosper in the future even more than it has prospered in the past, and that the individual scientific societies will not cease to cooperate with it.

If it is agreed that there are real benefits to be derived from an association of many distinct scientific societies, it will equally be granted that some advantages may be obtained from a meeting where so many points of view, modes of thought, and methods of investigation are represented as among the members of Section A. These diverse, and in some cases conflicting, points of view have arisen from the narrow specialization of recent times, and from the fact that the bounds of our knowledge have extended more rapidly than our individual capacities for encompassing them. Historically, astronomy and mathematics have been most intimately related. In antiquity the roll of the celebrated masters in one of these sciences would almost exactly coincide with that in the other; and in more modern times the names of Newton, Euler, Lagrange, Laplace, Gauss, Cauchy, Poincaré and many others, which are equally honored by both astronomers and mathematicians, show how much they still have in common. It is for this reason that in accepting our secretary's invitation to present a paper before you I have chosen to lay emphasis on these close relations; and in doing so I apologize only because I am so poorly prepared adequately to treat so worthy a theme. If you will permit the interjection of a purely personal remark, I should like to state that it is not solely because of the close relations of astronomy and mathematics that I have chosen my topic, but because a most fortunate experience has taught me that the lack of sympathy, if not of respect, for the efforts in one domain by workers in another is due almost wholly to a lack of acquaintance with them. I refer to the fact that for fifteen years I have been intimately associated with one of the former presidents of the American Association, with the present president, and with the vice-president of Section A. The first has taught me how the mind of the naturalist works, how vivid and constructive is his imagination, how fertile he is in inventing hypotheses, how exhaustively he gathers his data, and how impartially he weighs evidence where prejudices easily might have influence; the second has shown me how keen are the intuitions of the physicist, how his quantitative estimates almost rival in accuracy mathematical calculations, and how his marvelous instruments increase the delicacy of our senses a million fold at a single leap, making it unnecessary for us to wait inconceivable ages for their evolution to that degree of perfection; the third has revealed to me how beautiful are the logical structures which can be built on independent and consistent hypotheses, how keen are the pleasures in establishing all the harmonious relations involved in a mathematical theory, and how great is the satisfaction in the discovery of that which is common and fundamental to many apparently distinct theories. Here I confess my profound and equal respect for all these phases of scientific activity, and my belief that they are all of fundamental importance. If in my attempts to accomplish something I follow more nearly along the line of one than another, it is because I believe that on account of personal tastes and training I am less likely to fail there than in the others.

There does not seem to be a disposition on the part of any one to limit the field of activity of the astronomer. He is supposed not only to know how to measure the distances, to calculate the motions, and to determine the composition of the heavenly bodies, but also to understand fully those complex factors which produce the weather changes, to be familiar with certain mysterious forces which bring good or bad luck to an individual, to have reliable data respecting the location of heaven and its antithesis, and to be an expert on the questions of the freedom of the will, the existence of an infinite being, and the immortality of the soul. But the matter is quite different in the case of the mathematician. Often he is criticized for busying himself with pure fictions of the mind rather than with the so-called actualities of physical problems. It is no mere passive antagonism, for there are many places in these halls to-day where a storm can be raised by starting a discussion of imaginary numbers, hyperspace, or the non-euclidean geometries. It is easy to find men who will mark out the regions within which mathematicians should exercise their powers. It is an interesting psychological phenomenon that a specialist who has spent many years on a subject and has become a recognized authority in it seldom, if ever, will make any definite and general statement in regard to it; yet often he will not hesitate to make sweeping dogmatic assertions respecting things entirely outside his line, for example (to use a harmless illustration), respecting the merits of the tariff or the crime of seventy-three. Even those who have been expert in mathematics have differed much among themselves respecting what should be its highest aims. Fourier, in reporting on the work of Jacobi to the Academy of Sciences, said that natural philosophy should be the principal object of the meditations of mathematicians. In the introduction to his theory of heat referring to analysis he wrote "there could not be a language more universal and more simple, more exempt from errors and obscurities, that is to say, more worthy of expressing the invariable relations of natural objects. Considered from this point of view, it is coextensive with nature itself; it defines all the sensible relations, measures the times, the spaces, the forces, the temperatures; this difficult science is formed slowly, but it retains all the principles it has once acquired. It grows and becomes more certain without limit in the midst of so many errors of the human mind." Replying to the reproach of Fourier, Jacobi, in a letter to Legendre, said: "It is true that M. Fourier had the opinion that the principal end of mathematics was the public utility and the explanation of natural phenomena; but such a philosopher as he is should have known that the unique end of science is the honor of the human mind, and that from this point of view a question of number is as important as a question of the system of the world." Gauss agreed, for he said that mathematics is the queen of the sciences, and that arithmetic is the queen of mathematics.

Obviously it is just that the astronomer should allow the mathematician all the latitude in defining the limits of mathematics that he himself would desire if he were permitted to mark out the borders of the field of astronomy. It would be considered unwarrantable interference and an evidence of hopeless ignorance if any group of men should attempt to make astronomers confine themselves to those phases of their subject which are immediately useful to a busy world. If astronomy were limited simply to those parts which are necessary for time service on the land and the use of navigators on the sea; if it were necessary to abandon those mathematical theories of the motions of the planets and satellites which are in all respects the most perfect examples in natural science of harmony between theory and observation; if it were no longer permitted to use our powerful instruments in observing the peculiarities of the planets and the sun; if we were compelled to discontinue our investigation as to their origin and evolution; if we were under obligations to give up the spectroscope forever; and if we were forced to forego all further attempts to sound the almost boundless depths of the sidereal system and unravel its mysteries; if astronomy were put under these restraints, I say, then most of those incentives which in all the history of astronomical science have produced the rarest examples of devotion to ideals and the pursuit of knowledge would be removed. Astronomers do not admit the right of a partially informed world to prescribe the boundaries for their activities, nor do they in turn feel qualified or even inclined to impose any limitations upon the mathematicians. They could not even say what kinds of mathematics will be of use to themselves or to other branches of physical science. To take an example old enough to be understood in correct perspective, the interval between the discovery of the properties of the conic sections by Menæchmus and their first practical use by Kepler was 2,000 years, or more than nine times that which separates us from Newton. Astronomers will admit, then, that if the sole purpose of mathematics were to serve the other sciences, it would not be safe to circumscribe it by any boundaries. And most of them, I think, will go much further and join me in the sentiment that mathematics, altogether apart from its uses in other subjects, has a right to exist; that it is a part of the universe of ideas which to a thinking being is no less real and important than the physical universe; that its proportions and its symmetries which find perfect expression in its wonderful symbolism are, in satisfying the esthetic tastes, on a level with the fine arts; and that the process of drawing its conclusions calls for an exercise of the best and highest faculties we possess. If we were required to describe the proper field of mathematics we might say simply that it includes at least all that which all mathematicians together claim belongs to it.

Having admitted the breadth of mathematics, we have to consider what part of it has had at least its initial inspiration in astronomy. It might, perhaps, be argued with a good deal of justice that all of mathematics has originated directly or indirectly in the experience of the human race; that our capacity for those particular modes of thought which are essential to its development have evolved under the stimulus of the physical world. It is significant, at any rate, that there is such wonderful harmony between the results obtained by mathematical processes and our experiences. But it is not the purpose here to make any such claims, or to become involved in the difficulties of metaphysical discussions. No thesis has been laid down which it is necessary to defend, and no claim that astronomy has had an important influence on mathematics will be filed except where the evidence is perfectly clear and conclusive.

It was noted in the beginning that in ancient times the astronomers were almost invariably also mathematicians and reversely, and consequently that it is difficult to separate the two sciences of that time so as to determine exactly the influence which each had on the other. But there is one case in which the demands of astronomical problems certainly stimulated the development of a mathematical theory. Trigonometry was invented by Hipparchus, who was the most eminent Greek astronomer, both as a practical observer and as a mathematician. He determined the length of the year correctly to within six minutes of its true value, the obliquity of the ecliptic to within five minutes of arc, the annual precession of the equinoxes to within nine seconds of arc, the distance of the moon to within one per cent. of its value, the mean motions of the sun, moon and known planets, the changes in the moon's orbit, he made a catalogue of the fixed stars, etc. There is every reason to believe that these astronomical problems were those in which he was chiefly interested, and they made it necessary for him to develop trigonometry, and especially spherical trigonometry. His work was completed by Gauss nearly 2,000 years later in connection with the solution of the same problems.

We shall not, however, get any comprehensive view of the relations of astronomy and mathematics by citing, without some classification, isolated examples where the latter is indebted to the former. Such a procedure will give us no idea of the reasons for any of the great movements in mathematical thought. Moreover, the mathematical theories are so interwoven that it is difficult to pick out individual branches and to discuss their origins without being at least very incomplete. Therefore we shall content ourselves with broad classifications of mathematics, and to statements, with illustrations, of those parts where the practical problems of astronomy have had important influences.

Mathematics may first be divided into the metrical and the non-metrical branches. The former are vastly more important than the latter; or, since it is perhaps wise to avoid passing judgment as to what is important, the metrical branches have at least an enormously greater literature than the non-metrical. Recognizing the fact that there is much of a non-metrical character in those subjects which are regarded as being on the whole metrical in nature, and not wishing to insist on the possibility of making an absolute division on this principle, an examination of the Royal Society Index covering publications in mathematics from 1800 to 1900 shows that probably not one part in forty has been devoted to non-metrical mathematics. While there are certain non-metrical aspects of some astronomical problems, it would not be fair to claim that astronomy has had any essential part in inspiring these branches of mathematics.

Now considering only the metrical branches of mathematics, we may divide them into the mathematics of the continuous and the mathematics of the discrete. Here the problem of actually effecting the division is even more difficult than in the preceding case, for there is more intermingling and there seem to be more debatable points. In spite of these difficulties the mode of division is attached to certain fundamental characteristics either of the subject matter or of the processes employed. The ordinary theory of numbers is an example of the mathematics of the discrete. The theory of ordinary equations, for example, linear equations, may be considered as an example of the mathematics of the discrete or the continuous, according as the coefficients are regarded as discrete numbers or continuous functions of certain parameters. In such cases where the ideas of continuity are not essential to the formulation and treatment of the problems, they will be considered as belonging to the mathematics of the discrete. All those branches of mathematics in which continuity is an essential feature, as, for example, those involving derivatives, constitute the mathematics of the continuous.

On the whole, the problems of astronomy have not given rise to the mathematics of the discrete. While the physical universe seems to be made out of discrete things—atoms, corpuscles, units of electricity—it changes continuously from one state to another. Since a large part of the problems of the natural sciences relate to changes of position or state, such as the motion of a world or the evolution of an animal, this continuity is forced into the foreground in the applications of mathematics to physical questions. Consequently, in seeking for places where astronomy has made real contributions to mathematical theory we may restrict our search to the mathematics of the continuous. If there are no other subjects which have made similar contributions, we have at once the answer to the question of the extent of its influences. Since astronomy is more thousands of years old than most of the other natural sciences are centuries, it has naturally called forth most of those mathematical processes which have been needed in the others. About the only other natural science which has given rise to important mathematical theories is physics, which has forced attention to certain classes of partial differential equations and to the statistical methods employed in the kinetic theory of gases. Another important advantage astronomy has enjoyed is the delicate character of many of its observations and the high degree of precision of many of its theories. These have naturally directed attention to the questions of logical rigor. It is probably known to most of the members of this section that the numerically most perfect theory in all the range of physical science in all time is the lunar theory of our retiring vice-president. But the mathematics of the continuous has not been inspired by astronomy alone, or even by all the physical sciences together. In geometry the questions of tangents and areas have involved the same principles and have given rise to some of the same methods. Consequently we can conclude only that the problems of astronomy have given rise to some of the theories of the mathematics of the continuous.

It will perhaps be worth while to descend for a few minutes from the general to the particular, and to consider more concretely what contributions astronomy has actually made to mathematics. It is agreed by all that the invention of the calculus was one of the most important steps ever made in mathematics. It was founded first by Newton and a little later independently by Leibnitz. The work of either was sufficient to open the way to all that which has followed the invention of this important branch of mathematics. Newton's ideas were largely inspired by the consideration of physical phenomena, as is shown by the terminology and notation he used as well as by the problems to which he applied his methods. He spoke of *fluents* and *fluxions* and used the time as the independent variable, though he knew this was not essential. It simply indicates the stimulus of his ideas. On the other hand, Leibnitz used the terminology of geometry and seemed to have arrived at his ideas of derivatives through the consideration of tangents to curves. These differences constitute an internal evidence of the independence of the work of Newton and of Leibnitz.

The history of the application of the calculus in the century following its discovery constitutes one of the most glorious records of the achievements of the human mind. Mathematicians had a new method of enormous power and the greatest generality, while the laws of motion and the law of gravitation were the keys that unlocked a new universe to them. The work of Clairaut, d'Alembert, Euler, Lagrange and Laplace was one succession of triumphs. With the enthusiasm of explorers they traversed the worlds Newton and Leibnitz had opened, and with Laplace it was supposed the discoveries in them were about exhausted. The point to be emphasized here is that whatever may have been the origin of the calculus, its evolution into the larger domain of analysis in the century following its invention was due almost entirely to the stimulus of physical, and in particular astronomical, problems. There does not seem room for doubt that the very important place which analysis now occupies in mathematics is to a large extent due to its applications to astronomy.

Astronomy not only turned the attention of mathematicians to analysis, but it often determined the precise form their theories should take. Consider, for example, analytic differential equations. There are five distinct methods of developing their solutions—as power series in the independent variable, as power series in parameters, as limits of equations of finite differences, by successive approximations, and by successive applications of the variation of constants—all of which were devised under the pressure of practical astronomical problems and were applied successfully many years before the conditions of their legitimacy were fully established by mathematical methods. A more recent example is Hill's treatment of the linear differential equation having simply periodic coefficients, the properties of whose solutions were inferred by him from the properties of the motion of the moon. The problems connected with an infinite number of simultaneous homogeneous linear equations also arose in Hill's lunar theory. Poincaré's researches in the problem of three bodies led him to the discovery of many new properties of the solutions of differential equations. The question of the legitimacy of the series used in celestial mechanics, particularly when applied for long intervals, has forced a consideration of the problem of determining what classes of divergent series may be used and what conclusions may be drawn from them; and the same question has stimulated investigations of other modes of representing solutions, particularly as sums of polynomials in the independent variable, having wider domains of validity. In this direction Painlevé has achieved the most important results, and has shown how to construct functions which represent the solution of the general problem of n bodies so long as there are no collisions. If the forces were repulsive instead of attractive the developments would be valid indefinitely. But as Laplace said "nature does not care for analytical difficulties"; in fact, it fills the way of the mathematician with them. As a partial recompense for the difficulties it raises it often suggests methods for overcoming them, and these methods being made general in the symbolism of mathematics constitute new processes often applicable in many other directions.

One of the recent movements in mathematics is in the application of the postulational method. It consists in postulating the existence of certain elements which are wholly without properties except as they are imposed by the postulates and the explicitly stated axioms. The postulates and their implications constitute the theory. It is not to be supposed that the postulates are laid down at random, or even on any simple principle of their individual and separate characteristics. The sole guide is that taken together they shall yield as consequences certain relations which are in advance in the consciousness of the investigator; the additional implications are the contributions which the developed theory makes. I do not know why there has sprung up the recent interest in this method, but it is fundamentally the method used in natural science. The experiences are the certainties given in advance which must be implications of the theories. The atoms, corpuscles, units of electricity, etc., are the postulated elements. The theories are the postulated relations among the elements. If we let $a_1, \cdots, a_n$ represent the experiences, $x_1, \cdots, x_m$ the postulated elements, then we shall have

$$f_i(x_j) = a_i,$$

where the f_i are the theories. If one of these relations fails to hold it is necessary to modify the x_j, or the f_i, or both, and it is easy to cite examples from the history of science illustrating all these possibilities. The recognition of the fundamental identity of the process of constructing theories in the realms of natural science and of developing mathematics by the postulational method will undoubtedly be of great value to the former in showing what is really essential, and to the latter in inspiring almost endless points of view.

It is not necessary to cite more examples to show that mathematics owes much to astronomy, especially in the field of analysis. If it were proper to strike a balance it could probably be shown that the debt has been more than repaid, but in these unselfish sciences the privileges of foreign service are cherished as much as the treasures of domestic achievements, and therefore we content ourselves with the recognition of the interrelations.

In closing I may point out the truism that these interrelations are not limited to astronomy and mathematics. It is to the glory of astronomy that in it were initiated the two most fundamental intellectual movements in the history of mankind, viz., the establishment of the possibility of science and of the doctrine of evolution. Our intellectual ancestors in the valleys of the Euphrates and the Nile and on the hills of Greece looked up into the sky at night and saw order there and not chaos. By painstaking observations and calculations they discovered the relatively simple laws of the motions of the heavenly bodies, whose invariable and exact fufilment led to the belief that the whole universe in all its parts is orderly and that science is possible. In the modern world this conclusion is so commonplace that its immense value is apt to be overlooked, but a study of the superstitions and the hopeless stagnation of those portions of mankind which have not yet made the discovery gives us some measure of its worth. The modern supplement to the conception that the universe is not a chaos is that not only is it an orderly universe at any instant, but that it changes from one state to another in a continuous and orderly fashion. This doctrine that science is extensive in time, as well as in space, is the fundamental element in the theory of evolution and the completion of the conception of science itself. The ideas of evolution in a scientific form were first applied to the relatively simple celestial phenomena. More than a century before the appearance of Darwin's "Origin of Species," and the philosophical writings of Spencer, another Englishman, Thomas Wright, published a book on the origin of worlds. Laplace's nebular hypothesis gave the geologists a basis for their work, which in turn paved the way for that of Darwin. For half a century now the doctrine of evolution has been a fundamental factor in the elaboration of all scientific theories, and its influence has spread to every field of intellectual effort. It has been the good fortune of mankind that his skies have sometimes been free of clouds and that he has been able to observe those relatively simple yet majestic and impersonal celestial phenomena which have not only led to so important results as the founding of science and the doctrine of evolution, but have strongly colored his poetry, philosophy and religion, and have stimulated him to the elaboration of some of his most profound mathematical theories.

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STATISTICS OF GERMAN UNIVERSITIES

The twenty-one German universities show an enrollment for the winter semester of 1910–11 of 54,822 students, as against 52,407 students last winter. During the past five years there has been an increase in registration of no less than 12,432 students. Curiously enough, the winter enrollment exhibits a decrease, although only of a few students, against the previous summer semester, in which