SCIENCE

FRIDAY, JANUARY 20, 1911

CONTENTS

MSS. intended for publication and books, etc., intended for review should be sent to the Editor of SCIENCE, Garrison-on-Hudson, N. Y.

THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

THE RELATIONS BETWEEN JUPITER AND THE ASTEROIDS [1]

THE story of the smaller bodies which form part of the solar system belongs altogether to the nineteenth and twentieth centuries. The discovery of Ceres on the first of January, 1801, followed by those of Pallas, Juno and Vesta, in the next six years, gave promise of a new field for the astronomer. Nearly forty years, however, elapsed before any more were found. The improvements in star maps about the middle of the century enabled observers to detect new objects more easily, so that from 1845 to 1860 over sixty were captured. Since that time scarcely a month but brings one or more to the list, and now nearly 700 are known. But this by no means completes the tale. New asteroids are constantly being recorded and receive a temporary designation and are perhaps observed two or three times. In fact the number of new discoveries has become too great for the few astronomers interested to obtain orbits of sufficient accuracy for future observation. It has become a question whether the search should be continued, and if so what plan should be adopted for the computation of the orbits.

To the student of celestial mechanics these small bodies furnish many interesting problems. The older planets and our own moon have had such thorough attention accorded to the study of their motions, that the outstanding difficulties are almost solely in the last refinements and

[1] Address by the retiring vice-president of Section A, American Association for the Advancement of Science, Minneapolis, December 28, 1910.

the most minute consequences of the laws under which they run their courses. Their orbits differ but little from circles, and for practical purposes the series of terms which represent their motions are easily obtained, even if the process of so doing may sometimes be a lengthy one. Far otherwise is it with the asteroids. Many of their orbits are highly eccentric and inclined at large angles to the plane near to which the large planets circulate. But the most interesting problems are those which arise from the near presence of the second largest member of the solar system, Jupiter, whose mass is but little less than one thousandth that of the sun.

Jupiter, great as it is compared with the other planets, is yet small relatively to the central body which mainly controls their movements. In general its average effect on a body which does not come very near to it must be small. Under certain circumstances, it may cause very considerable deviations for a time, even in planets which do not approach it very closely. Astronomers have generally divided the disturbances of the motion of one body produced by another into two classes, periodic and secular. The latter are properly those which change the motion always in the same sense so that they would ultimately cause a change to another type of motion. As a matter of fact, however, the division is arbitrary and in a strict sense inaccurate. So far as we know, all the disturbances produced in our solar system by gravitation are really periodic, but some of the periods are so long, extending to thousands and ten thousands of years, that it is more convenient in the short space of time during which we desire to know the motion to treat certain of them as secular.

These long period deviations are often of very considerable extent. They may be divided into two classes, "proper" and "accidental." In the former all the bodies which have the same type of motion have the long period terms, for instance, the slow oscillations of the eccentricities of all the planets due to the attractions of one on another are of this type and they arise principally because the masses of the planets are small compared with the mass of the sun.

The accidental terms are those arising from a synchronism of periods. They should be regarded perhaps as a result of the defects of our mode of representing the motion in symbolic form. However this may be, their presence causes a real practical difficulty which must be solved. We regard the motion of each body as having a principal period of revolution round the sun in a circle, with deviations from circular motion due to the eccentricity of the orbit, these deviations having periods which are multiples of the principal period. Thus there will always be some period in the motion of one planet very near to a period in the motion of another planet. When the degree of approximation we require is settled there will only be a limited number of accidental corresponding periods. If the difference between the two periods is small, a term of long period arises, and the amplitude of the oscillation is nearly proportional to the long period (or to the period squared) and it might seem that when the period became infinitely great the amplitude would also tend to infinity. In physical terms the motion would be unstable. It is not so in general. When the amplitude begins to approach very large values the motion may still be stable. If it is, one of two things has occurred. Either the difficulty is a symbolic one, that is, our mode of representing the motion is defective for large oscillations, and the difficulty can be bridged by choosing some other analytical representation, or the type of the oscillation may completely change; when the latter happens, it is generally necessary to change also the symbolic representation.

A change in the character of an oscillation which does not cause real instability can be illustrated by a simple example. This illustration, while it has no immediate bearing on the question of synchronism, does show the change in the character of the motion which is sometimes produced by a synchronism.

Let us consider the motion of a rod, one end of which is pierced to admit a horizontal axle so that the rod can rotate in a vertical plane. Suppose that the rod is rotating so rapidly in one direction (called positive) that it makes many complete revolutions in a second. There will be slight differences of velocity at various points, differences which can be expressed quite accurately by a single harmonic term. Owing to frictional resistances the speed will gradually diminish. With diminished velocity the difference between the velocities at the highest and lowest points increases in a ratio which varies very nearly in the inverse ratio of the average speed. When the speed has so far decreased that the ratio of the velocity at the top to that at the bottom is very small, we can no longer express the differences of velocity approximately by a single harmonic term, but must include the higher harmonics. At the critical stage when the velocity at the top is just zero, the representation fails. From our knowledge of the physical side of the problem we know that this failure is not owing to the defects of the representation, but that it is due to a change in the character of the motion. The rod ceases to make complete revolutions and begins to oscillate to and fro with diminishing amplitude until it finally comes to rest at the lowest point. There is another side to this example which is important for what follows. The difference between the velocities at the highest and lowest points is continually increasing, but it does not become infinite. At the critical stage it reaches a maximum value and, becoming discontinuous at this point, has its range suddenly doubled. The minima and maxima are nearly equal in magnitude but of opposite signs. The range then diminishes until it reaches the limit zero. The critical point is, of course, a position of unstable equilibrium and then any small force which may be acting, but which could previously be neglected, may determine the character of the future motion.

If we neglect the resistance of the air and imagine that the axle on which the rod is mounted is made to turn in the opposite sense to the original rotation of the rod, the slight friction between the rod and its axle will gradually tend to stop the motion. But when the rod is nearly at rest close to its highest position one of two things will happen; either the rod will begin to oscillate as before, or after the first oscillation it will be carried past the highest point and begin to rotate in the same sense as the axle with increasing average velocity.

On the analytical side we have to notice mainly one point. Before the critical stage the angular motion is expressed by an angle which increases continuously with the time. As the motion gets slower the variations from uniform increase become more and more marked. After the critical stage is passed and the rod is oscillating to and fro, the angle itself varies between two limits which are less than 360° apart, finally settling down to a constant value. If, however, the rod begins to make complete revolutions in the opposite direction, the angle diminishes continuously with the time, and its variations from uniformity become smaller as the mean angular velocity increases negatively. It is true that we are accustomed to express the small oscillations of a pendulum by means of angles which increase uniformly with the time, but it is the angle of oscillation which is so expressed and this angle varies between finite limits.

I must apologize for insisting at such length on an elementary illustration with which you are all familiar. Though it serves to make clear the types of deviations which Jupiter may induce in the motion of an asteroid, the analogy must not be pushed too far. It fails to illustrate what may happen to a planet when the stage which corresponds to the rod at rest near the lowest point of the circle has been passed. And it takes no account of the numerous short period changes which the planet experiences and which can not be altogether neglected. Moreover, forces may be present which may tend to increase the oscillatory motion, so that after a period of hesitation the planet may be compelled to settle down into a motion which corresponds to a rotation of the rod in the negative direction, and this independently of the minute forces which may determine its motion just after the velocity has first become zero.

These periodic changes, in which the variable is constrained to remain between finite limits, are generally known as librations. The familiar example furnished by the motion of the moon relative to its center of mass is of course a result of the synchronism of its period of rotation about its axis with that about the earth. The delicate balancing of the conditions necessary for a libration makes the term unusually appropriate in the cases of the motions of the asteroids.

All the earlier discoveries indicated that the smaller members of the solar system, which were not satellites of some planet, occupied a more or less well-defined region between the orbits of Mars and Jupiter. But the distribution was by no means continuous. It was soon seen that if the mean periods as then determined were arranged in order of magnitude there were gaps in the list which could not be explained by the law of averages. It was, I believe, Professor Kirkwood, of Indiana, who first pointed out that these gaps existed at or near the places where the periods are commensurable with the period of Jupiter. With the increasing number of planets discovered the discontinuities have become more accentuated. In general, the smaller the two whole numbers which represent the ratio, the wider is the gap. It has been known that such cases of commensurability produced great difficulties in computation and that the ordinary methods failed. The idea was this—near commensurability of a forced period and a natural period produces large deviations, growing larger the more closely the whole number ratio was approached. It was thought that when the ratio became exact the deviation would be infinite, or in physical language the orbit was unstable. But the deviation, at any rate so far as at present known, does not tend to an infinite limit. There appears to be a maximum value which can not be exceeded, and if the planet is so started that this maximum is passed, the ratio of the two periods becomes exact and oscillations about this exact ratio occur. The natural period becomes a forced period. The analogy to the case of the rod is evident.

As early as 1812 Bessel had pointed out that the periods of Jupiter and Pallas are very nearly in the ratio of 7 to 18 and that the attraction of Jupiter must maintain this exactly, that is, the deviations from it would be oscillatory and not secular. Newcomb invoked the same idea to combat the opinion that instability was the cause of the gaps in the distribution of the asteroids. In a word, the oscillations become librational and have finite limits, or in physical terms, the motion is stable. Thus instead

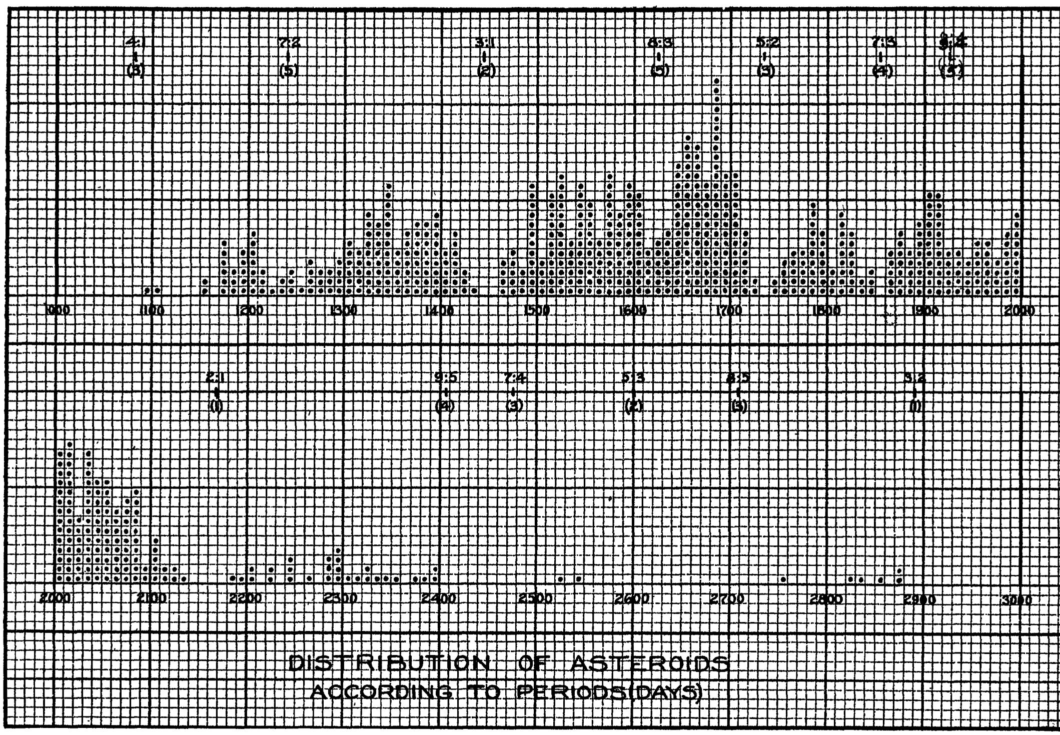

FIG. 1

of causing an absence of asteroids at the positions of commensurability, with this idea the action of Jupiter should rather tend to increase the number, since a definite proportion of those very near would be compelled to librate about the exact ratio. It is not clear to what extent the published elements are dependent on the terms of very long periods, but the main issue is not affected; none of the known asteroids in the principal lacunæ appear to librate.

In order that you may see the extent of the inequalities in the distribution I shall throw upon the screen a chart (Fig. 1) in which the abscissæ represent the mean periods in days and the ordinates the number of planets with those periods. Each vertical division represents ten days and the number of planets whose periods fall within any division is shown by the height of the curve at that place. There are about 630 asteroids represented. The curve falls to zero at 2,160, 1,850, 1,730 and 1,440 days and there are less well marked minima in other places. Now these four minima include the places where the periods are in the respective ratios of 2:1, 7:3, 5:2, 3:1 to the period of Jupiter. There are also indications of minima at the places where the ratios are 8:3, 7:2, 9:4. These are all the whole number ratios within the range with a denominator less than 5.

I am going to speak more particularly

of the gap at the ratio 2:1, this being the most extended of all. Here there are no known planets with periods between 2,120 and 2,204 days; the exact ratio is 2,166, not very far from half way between the two sides of the gap. The comparatively few planets on the outer side of the gap may be a consequence of the general distribution, but the sudden increase in the number on the inner side suggests that other causes have been at work.

In attempting to give an explanation of the phenomena by gravitational forces alone, I must on this occasion omit any mathematical discussion and simply lay before you the main results at which I have arrived. Until this discussion is published in detail, you will naturally reserve your opinions as to the validity of the argument. The reasons for presenting the results in advance of the methods will be evident in the course of my remarks.

The motion of an asteroid near the critical places depends mainly on two quantities, the apparent eccentricity of the planet at any time and the difference between the actual period and the period which is twice that of Jupiter. In the theory both these are quantities such that if at any moment the attraction of Jupiter were suddenly annihilated, the asteroid would continue its motion in an elliptic orbit round the sun with this period and eccentricity. They are thus determined by the position and velocity of the planet at the moment. The attraction of Jupiter causes the temporary period and eccentricity to vary and it is the variations of those two quantities that are the main factors. Now the equations which I have obtained give the principal parts of these two quantities in the form of the square roots of variable functions. In order that they may be real these variable functions must always be positive. One of two things must happen. Either the variability of the functions is forced, that is, it is independent of the motion of the asteroid, or it is free. In the latter case the motion of the asteroid is limited. Under certain general conditions it has long been known that the equation giving the period is to a large extent free so that the asteroid may librate or not, according to the values of the constants entering into the equation, and there is nothing to prevent the constants from having such values. But the period also depends on the eccentricity, and the possible variations of this quantity can not be neglected.

The eccentricity and the difference of the actual period from that of Jupiter are really connected by two equations which can not be independently treated. The eccentricity is also dependent on the short period terms which from this point of view may be considered as of forced period, since the periods of the larger terms are very nearly multiples of the period of Jupiter. I find from this that the eccentricity can not permanently remain below a certain limit and consequently that if a libration of the period about the critical ratio occurs, it can not have a very small amplitude. Referring to the analogy of the rod, we discover the presence of forces which prevent the angle of oscillation of the rod to and fro from descending below a certain finite value. The oscillation can not become infinitely small. This result is in agreement with a theorem proved by Poincaré that no periodic orbit at such a critical place exists.

Reducing to numbers, I find that the lower limit of the slow variation of the eccentricity for an asteroid just before it reaches the critical stage between libration and non-libration is about one twentieth and that at some time it must become as large as one seventh. If there is an asteroid which has passed this stage and is librating, the upper limit of the eccentricity is at least one fifth. Thus at some time the asteroid must have a comparatively large eccentricity. These large changes in the eccentricity take place very slowly. Hence the asteroid will during one interval perform many revolutions round the sun in a nearly circular orbit and during another interval many revolutions in a quite eccentric orbit.

It has been mentioned that these large variations of the eccentricity are of very long period—the same as that of the libration—and that they increase with the extent of the libration. One may regard the asteroid as slowly extracting energy from Jupiter (or losing it to Jupiter) until it has accumulated (or lost) a maximum amount, when the reverse process takes place. Jupiter, owing to its immense mass in comparison with that of the asteroid, will not have its motion sensibly altered by the loss or gain. If the type of motion changes when this loss or gain is near a maximum, the asteroid may in future so move that the energy never returns to its original source. The type must then be unstable.

In my discussion all powers of the eccentricity but the lowest have been neglected. This omission prevents us from obtaining a proper representation of the motion when the eccentricity has risen to so large a value as one fifth, and it is to be remembered that this is the lowest maximum which any librating asteroid can have.

If we take into account higher powers of the eccentricity, it is seen that the extent of the variation must be increased to a considerable extent. I have not so far been able to obtain numerical results beyond this stage. The indications are that the eccentricity increases so much at some time that the stability of this type of motion is threatened and that the asteroid is compelled to take up some other type which is more stable.

Although the facts I have stated do not furnish a proof that librations round the ratio 2:1 can not exist, they at least indicate the direction in which stability becomes doubtful. The usual analytical representation, which is possible for moderate librations of the period, fails where the eccentricity becomes very small or great, and the failure is partly due to the short period terms. The question which requires solution is whether the failure with large eccentricities is merely a defect of the mode of representation, or whether it is due to actual instability. My own impression is that the limits of stability of libration about the principal ratios are so near together, that the chance of an asteroid lying between them is very small. This impression is, however, not independent of the observations. There are no known asteroids which perform librations round the ratio 2:1 and there are none which approach libration very closely. The lacuna in the distribution doubtless admits of other interpretations, but the framing of hypotheses to explain it must properly await a more complete discussion of the consequences of the law of gravitation.

The distribution of the asteroids on either side of the gap furnishes certain tests. The theory indicates that asteroids which initially had large eccentricities are more likely to librate than those with small eccentricities. If the large eccentricities, as I believe, tend to become unstable, it follows that the gap must be wider for them than for the small eccentricities. The next slide (Fig. 2) shows the distribution of eccentricities greater and less than one fifth, round the ratio 2:1. It shows also the distribution by differences of .05 of the eccentricities above .20. The tendency of the asteroids to be more distant from the gap as the eccentricities increase is very noticeable.

Again, according to the theory of a number of asteroids started at various times with the same eccentricity and with periods which bring them near the ratio on the inner side but not such as to allow them to librate, we shall at a future time be more likely to find the values of the eccentricity large at a distance from the gap and small near the gap. The reverse would occur with asteroids on the outer side. The observations confirm this result in a marked degree on the inner side. The number of known asteroids on the outer side is too small to furnish a test, but if the observed distribution is typical of the real distribution the theory would be confirmed. The need of more asteroids, especially on the outer side, is evident.

That these peculiarities are not consequences of the general distribution of the large eccentricities will appear from the next slide (Fig. 3). It will be seen that there is a notable deficiency near to the gap and that this deficiency is much more marked than the general distribution would lead one to expect.

Further support is derived from the inclinations. With the ratio 2:1, there should be no very noticeable difference in the distribution of large and small inclinations round this gap, and the next slide (Fig. 4) shows this. The curves for i greater than 10° can be compared with those for i less than 10° and also with those for each 5° greater than 10°. It is seen at once that there is no selection of the large or small inclinations. The distribution for large inclinations appears to follow the same general law as the distribution for small inclinations. If we refer back for a moment to the slide giving the distribution of the eccentricities, it will be seen how different the two distributions are.

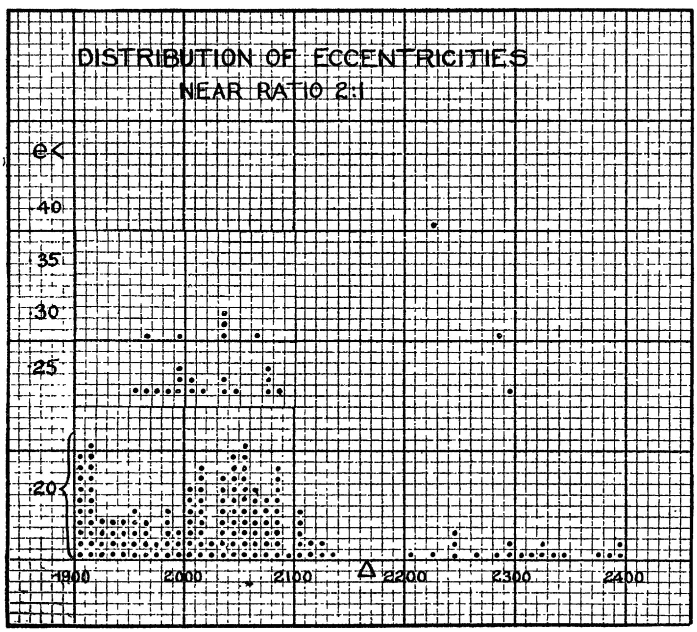

Fig. 2

I have hitherto spoken solely of the ratio

2:1. Similar results can be obtained concerning the distribution round the other ratios, but these I must dismiss in a few words, mainly because the details have not yet been worked out. It can be stated, however, that the inclination can not in general be neglected when the difference between the two terms of the ratio exceeds unity. Nevertheless, the general result holds that larger eccentricities are more likely to be absent very close to the gaps than smaller ones.

Referring again to the chart of the general distribution of the eccentricities (Fig. 3), we notice that no one of the ratios, even when the difference of the two terms is 5 (corresponding to a libration with a coefficient depending on the fifth power of the eccentricity), has a close asteroid with an eccentricity greater than one fifth. This is nowhere more strikingly illustrated than at the values 7:3 and 9:4. There is a large maximum of small eccentricities between these ratios, and only three asteroids with eccentricities so great as twenty-five in the same space. A similar phenomenon is observable between the ratios 7:2 and 4:1.

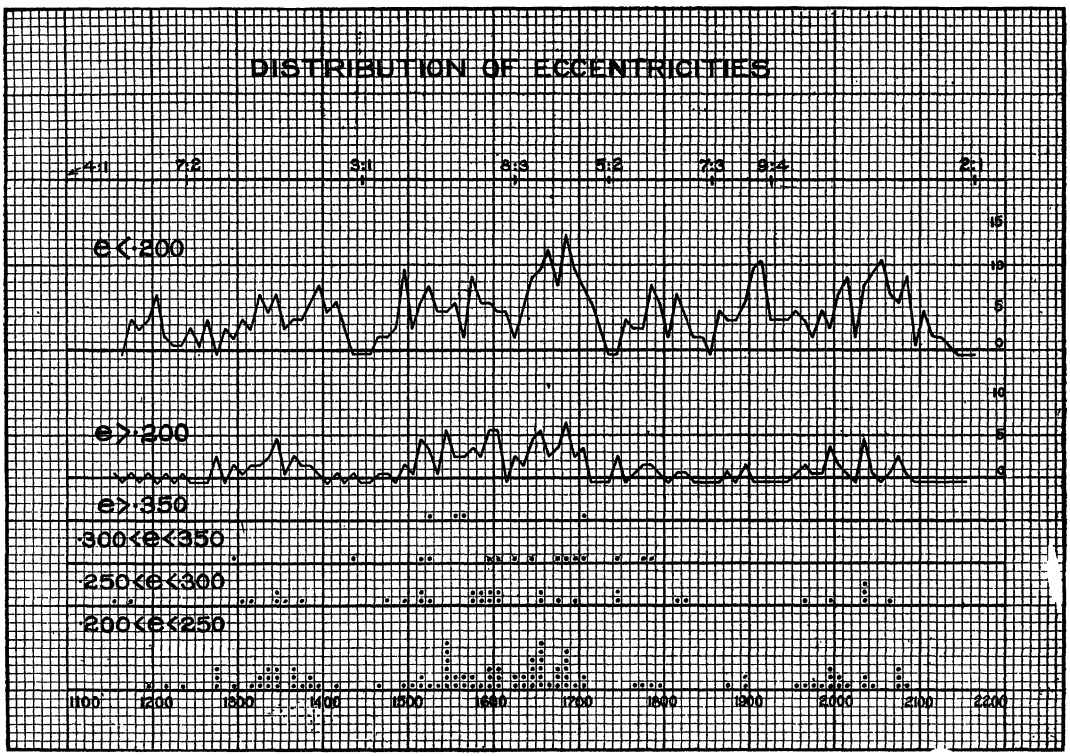

FIG. 3

The application of the theory to the satellites and rings of Saturn, though not within the title of my address, can not well be passed over. It is now known that the mean periods of the revolutions of Titan and Hyperion round Saturn are exactly in the ratio of 4:3 and that the actual periods librate about this ratio. The case is quite similar to that of a ratio 2:1. But here the relative disturbing force is much smaller and the variations of the eccentricity due to this libration are also much smaller. There is, in consequence, less tendency toward instability due to this cause and in fact the numbers do not indicate such a phenomenon. The eccentricity does not rise very greatly in value.

The divisions of the ring system admit of an explanation if we suppose them to consist of clouds of small bodies subject mainly to the attraction of Saturn and of its satellites. A body on or near a critical radius will experience considerable variations of eccentricity which are of long period. A narrow ring of these bodies will therefore be moving in ellipses of different eccentricities which vary for the different bodies much more than for bodies not near this ring. Hence the range of distance of the bodies composing the narrow ring will be spread out much more than for other rings of the same width. At any given time the number of large eccentricities will on the average be much greater than the number of small eccentricities. On both accounts the number of bodies visible at any one time will be least on the critical radius, rise to a maximum on each side of this radius, and then diminish to the average distribution. Thus we get a ring darker than the average between two rings brighter than the average—a result which tends by contrast to increase the apparent difference in illumination.

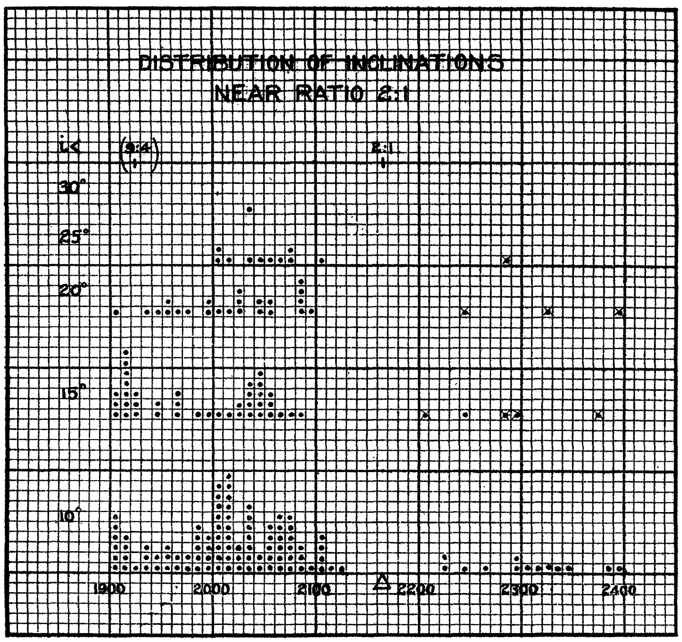

Fig. 4

Another result of this variation of eccentricity may be mentioned. Observation shows that the apparent form of the rings is very nearly circular. It would seem that the orbits of the separate bodies must have small eccentricities in order to avoid collisions in which the velocities differ considerably. Near a libration region the eccentricities can not remain very small. In the course of long periods of time it would result from this that encounters have taken place which changed the velocities of the colliding masses sufficiently to cause them to pass away from the region of libration. Such collisions may affect the motion in two ways. They may increase the extent of the libration so that the body passes outside of the limits within which librations are possible, or they may at the time diminish the extent of the librations so far that the lower limit of the eccentricity at which librations can be maintained is reached. In either case the body escapes from the region of libration and describes an orbit at an altered mean distance and with a period whose difference from the exact ratio never becomes zero. Thus while the apparent divisions of the ring are not directly caused by the instability of the librations, they are partly caused by perturbations of large amplitude and perhaps also by the results of collisions arising from those large perturbations.

I have hitherto spoken of the effect of Jupiter's attraction on certain groups of the asteroids. A little space must be devoted to the phenomena which are caused in the distribution of the perihelia and nodes of the approximate ellipses in which the asteroids move.

When the orbits of but fifty of the asteroids were known it was noticed that the positions of their perihelia tended to group themselves round the perihelion of Jupiter. A similar phenomenon appeared, though not in so marked a degree, with their nodes and the node of Jupiter. Newcomb proved that the groupings were consequences of the attraction of Jupiter. His explanation, briefly stated, amounts to this. Jupiter causes the perihelia of the asteroids to revolve very slowly round the sun; some of them will librate about the perihelion of Jupiter. In the former case the rates of motion of the lines which join them to the sun are least when these lines are crossing the perihelion of Jupiter, and most rapid when crossing the aphelion of Jupiter. Hence at any given time we are likely to observe more asteroids whose perihelia are near that of Jupiter than asteroids whose perihelia are in other positions. Asteroids whose perihelia librate must be comparatively few in number. The variations of the eccentricity follow the same law. With the nodes and inclinations the effect is less marked. In the latest published investigation by von Brunn, who followed Newcomb's methods with 400 bodies, the explanation is confirmed. The chart (Fig. 5) which is before you on the screen shows the extent of the variations. It is formed with 630 asteroids. The points joined by a continuous curve represent the numbers of asteroids whose perihelia lie in the twelve divisions into which the whole circumference has been divided. The points joined by a dotted line represent the average eccentricity of those asteroids whose perihelia lie in the corresponding divisions. Below are placed the letters *J, M, E, S,* representing the positions of the perihelia, and the letters *J′, M′, E′, S′,* representing the positions of the aphelia of the planets Jupiter, Mars, the earth and Saturn. The two upper charts show the results for the first 300 and the last 330 discoveries, and the lower chart the whole 630. The effect of Jupiter is quite clear. The traces of the other planets are doubtful. It is true that there is a distinct minimum of the perihelia at the group 10 which contains the aphelia of Saturn and the earth and that it is present in all three charts. But there is no corresponding rise at the perihelia, or at any rate the rise is doubtful.

The features of the distribution which are the same in the earlier and later discovered planets can, without much danger of error, be laid aside as consequences of the attractions of the sun and planets.

But there are certain differences which call for special notice. The minima of the perihelia and eccentricity which occurs in the group 7 with the first 300 asteroids occurs in should, of course, be diminished by one eleventh of its height, for strict comparison with the first chart, on account of the total numbers of planets used; but this does not

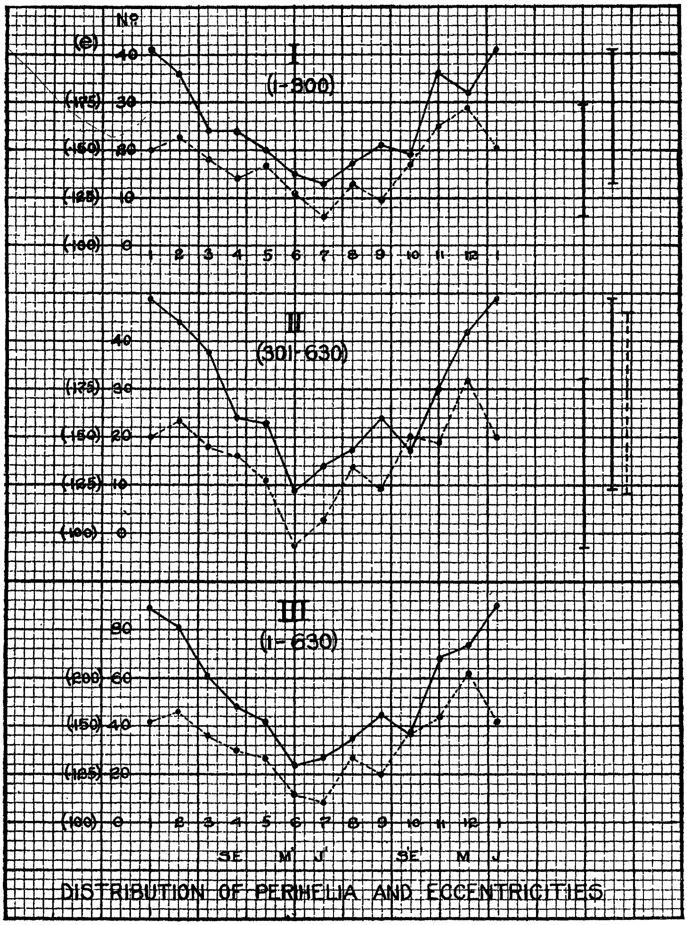

FIG. 5

the group 6 with the last 330 asteroids; the displacement does not appear to be accidental. But the most remarkable difference is the greater range between maximum and minimum with the later discovered and presumably smaller bodies with the earlier ones; this difference is observable in both the perihelia and eccentricity curves. Each ordinate of the curve which represents the perihelia in the second chart materially affect the observation. The range is greater by nearly 50 per cent. for the perihelia and by nearly 80 per cent. for the eccentricities. Is this a relic of the initial conditions or the effect of some force other than gravitation, for example, light pressure or the impact of material particles entering or leaving the solar system, in which the mass of the asteroids affected will play some part? There is here a wide field for speculation which I shall not attempt to enter this afternoon.

I have hitherto spoken of asteroids which belong to the main group. Others are scattered either singly or in small groups at quite different distances from the sun. Eros, whose orbit lies within that of Mars, is so well known owing to its value for the determination of the parallax of the sun, that it needs only a passing mention. A small group of five asteroids just inside the mean radius at which the period is two thirds that of Jupiter, is of interest as an illustration of the features of those asteroids the large perturbations of which depend on the first power of the eccentricity. As at the place where the ratio is 2:1, the planets are on the inner side of the gap. It is, of course, not possible to draw any conclusions from so small a number.

But the most remarkable illustrations of the problem of three bodies are four small asteroids which have been found within the last five years. Before that time the period and therefore the mean distance from the sun of every known asteroid were less than those of Jupiter. In February, 1906, Wolf observed a body whose velocity appeared to be unusually small and, in the course of a few observations, it was seen that its orbit could not be very distant from that of Jupiter. Charlier suggested that it might probably be an illustration of one of the two cases in which the problem of three bodies can be solved without approximation. Long ago Laplace proved that if three bodies be placed at the corners of an equilateral triangle and started revolving round their common center of mass with a properly adjusted angular velocity, they would continue to do so indefinitely. The triangle remains equilateral and each body describes an ellipse with the center of mass as a common focus. Further, it has been proved that, provided the masses satisfy certain relations, the motions are stable, that is, small oscillations about the triangular positions would not involve the ultimate loss of the type of motion; the triangle would always remain nearly equilateral. As the mass of the asteroid is so small that it may be neglected, the condition for the stability of small deviations is that one mass shall be at least twenty-five times the other; since the sun is over 1,000 times as heavy as Jupiter the condition is well satisfied. There seems to be no doubt that the theory is a correct explanation of the facts. In the intervening years three more asteroids with similar orbits have been discovered. Of the four now known, three are near the point 60° in advance of Jupiter, and one is near the point 60° behind. The mean distances of all four from the sun differ but little from that of Jupiter.

If we examine the orbits of these asteroids on the hypothesis mentioned, namely, that they are merely deviations from the triangular solutions, we have to admit at once that the deviations can not be considered very small. One of them may move as far as 17° away from the mean position as seen from the sun, and a question naturally arises as to the permanent stability of such large deviations. The periods of the oscillations must be very long, not less than 150 years, so that we can not hope to settle the question by direct observation.

These oscillations may be regarded as librational. Bodies well beyond the orbit of Jupiter will revolve around the sun more slowly than that planet, those inside more quickly, and between the two we get a region in which the mean period may be the same. There may then be librations about this mean period which carry the body first inside and then outside the orbit of Jupiter. Referring back to what has been said concerning the stability of librations around the ratio 2:1, we might be tempted to conclude that the stability of these new asteroids would be still more doubtful. This is not so. When the mean period is the same as that of Jupiter, the librations are very nearly independent of the eccentricity and inclination unless these are very large; and conversely, the latter are but little dependent on the librations of the mean period. It is the only case in which this independence occurs. The conditions governing these librations are so different that a separate investigation of them is necessary. I had hoped to lay before you the results which I have so far reached in determining the possible motions of asteroids which circulate with the same mean period as Jupiter. In order to do so it would be necessary to exceed greatly the time allotted to me. I must content myself with the remark that much larger deviations from the triangular positions appear to be possible and that these may become so extended that a single asteroid may go round both triangular points, passing from one to the other on the opposite side of the sun from Jupiter, and yet not at any time approach very closely to the planet. While describing such an orbit the distance of the asteroid from the sun would vary but little.

In conclusion, a few words must be said concerning the present condition of the observational material and its future needs. There are about 680 bodies whose orbits are more or less well defined. We have seen that the distribution is by no means uniform. With the number on hand we have perhaps sufficient material to determine the general law which governs the arrangement of the orbits. This law is, however, only one of the problems which the system presents. The student of celestial mechanics needs more asteroids in special regions. Reference has been made to the abundance of known orbits on the inner side of the gap which corresponds to twice the period of Jupiter and the lack of material on the outer side. Every asteroid added to those very near the gap will be welcome, but there is a real need for more on the outer side. In any case new bodies whose periods are close to this ratio should be observed until the eccentricity and inclination can be obtained with moderate accuracy. In the course of the search it is possible that one may be found which is within the narrow limits of stability (if the distance between these limits is not zero) of libration. At every opposition a few observations of all the known asteroids near the gap should be made, so that the orbits may be verified and the long period terms compared with theory. It will perhaps be advisable to limit the number of asteroids chosen for continuous observation. If so, a selection can be made which will give types of small, moderate and large inclinations and eccentricities.

A reference to the general chart of distribution of the mean periods will indicate what observational material is needed at the other gaps. Aside from the asteroids whose periods have the features which I have just described, every new discovery of a very large eccentricity of inclination should be retained. I would suggest that eccentricities above three tenths or inclinations to the orbit of Jupiter above 20° would perhaps not unduly tax the capacity of those observers who have made and are making valuable additions to our store of knowledge of these bodies.

Finally every asteroid whose period indicates that it is outside the main stream or near its edges should not be lost. More particularly the search might be directed towards the discovery of bodies having the same mean distance as Jupiter, but with librations more extended than those of the four at present known. As such bodies are moving most slowly when farthest from the triangular points the search would have greater chances of success within twenty-three degrees to thirty-five degrees of Jupiter on either side, or nearly in opposition to Jupiter. These three areas are nearest to the earth at different times of the year.

Another practical problem is presented by the computation of the orbits. This is mainly a matter of expense so far as the ordinary asteroid is concerned, and one way of meeting the difficulty was shown by Watson, who endowed those discovered by himself. We need not, however, demand that explorers should be responsible for looking after their own discoveries. The asteroids of each type of motion form a group which it would be most economical to treat together if at any time a fund were obtained for the development of the problem. For the present, ephemerides sufficient to identify each body will serve. When a reasonable number of accurate observations, extending over a considerable period of time, has been obtained, the comparison of the observations with theory will be of interest to students of celestial mechanics, and the rest of the work will take care of itself.

In particular, the Trojan group revolving at the same mean distance as Jupiter and at present consisting of the four bodies, Hector, Achilles, Patroclus and one as yet unnamed, will not suffer from neglect. They appear to show one, perhaps the main, stage of transition from bodies superior to the orbit of Jupiter to those inferior to that planet and possibly to those which have become his satellites. Their separate paths of motion are interesting to the mathematician, but even more so to the astronomer, since they appear to indicate a new set of periodic orbits in the problem of three bodies. The remarkable series of families of such orbits obtained by Sir George Darwin has shown how far such an investigation may lead. A single family may have several types. Those which I have described appear to belong to perhaps two families at the most, but nevertheless the extent of the work necessary to discuss them is very considerable, and the mathematical portion of it is by no means simple. They may be peculiar to systems like our own. The work of Sir George Darwin will doubtless have applications to multiple systems of stars, the observational material of which is being rapidly gathered by means of the spectroscope, the heliometer and photographic plate.

Theories as to the mode of formation and development of our solar system will, I believe, receive some assistance from these orbits of transition. If I have not touched on such questions it is not because the temptation to do so was not present. Such matters, however, require an extended discussion under various hypotheses. As Brodetski has shown, different hypotheses concerning the nature of possible resisting media lead to quite different results. My main object has been to attempt to set forth certain actual and possible types of motion within our solar system at the present time in such a manner as to indicate in what direction the theorist and the observer can best act in cooperation. The observer, now that the mass of accumulating material threatens to become too great for permanent record, needs assistance from the theorist so that he may direct his energies into the most useful channels, and not less does the theorist need the help of the observer so that his results may receive confirmation and that he may obtain suggestions for future investigations.

ERNEST W. BROWN