SPECIAL ARTICLES

NOTE ON THE FORMULAS FOR ENERGY STORED IN ELECTRIC AND MAGNETIC FIELDS

CONSIDER a charged sphere. Let it grow in size. The potential decreases for the same charge as the radius increases. Hence the potential energy also decreases. The tubes of force, everywhere pulling the surface out toward infinity, are losing the potential energy of their stretched condition, and at infinity they have closed up and the potential energy has disappeared from the potential state.

We may then consider the energy as residing, not in the sphere but in the dielectric outside, and that the amount of energy that disappears from the potential state at each step is entirely in the spherical shell of the dielectric, which makes up the difference in volume between the successive steps in the growth of the sphere. We have then, only to calculate the difference in potential energy for two slightly different radii of the sphere and divide by the volume of the spherical shell, and we shall have the density of the energy in the electric field. It is to be noted that the electric field at any point outside the sphere is unchanged by the growth of the sphere, since the number of tubes of force, and hence the amount of their crowding, depends only on the charge and not on the size of the sphere.

Let r be the radius of the sphere, v the volume, e the charge, E the electric field, ψ the potential, P the potential energy, ϵ the dielectric constant.

By definition ψ is the work necessary to carry unit charge from infinity to the sphere, or

$$\psi = \int_{\infty}^{r} E dr = \int_{\infty}^{r} (e/r^2) dr = e/r, \qquad (1)$$

which might have been written immediately, since the capacity of a sphere is r. Also by definition

$$E = d\psi/dr = d/dr(e/r) = e/r^2. \qquad (2)$$

We have also

$$P = \tfrac{1}{2}\psi e. \qquad (3)$$

From (1) and (3),

$$P = e^2/2r.$$

Differentiating, we get the change in potential energy due to a small change in radius,

$$dP = -e^2 dr/2r^2,$$

the negative sign meaning a decrease in energy for an increase in radius. The volume of the shell is $4\pi r^2 dr$, and the loss of potential energy per cm^3 is, by equation (2),

$$dP/dv = -e^2/8\pi r^4 = -E^2/8\pi.$$

Hence the energy in the dielectric is $E^2/8\pi$ ergs per cm^3.

If $\epsilon \neq 1$, the charge for the same ψ and the same E is greater and we have to write $\epsilon\psi$ and (ϵE) instead of ψ and E in equations (1) and (2), to make them hold numerically. This followed through gives, finally,

$$E(\epsilon E)/8\pi.$$

The expression for the energy in a magnetic field follows in exactly the same way; we have only to substitute m for e and H for E in the equations above. We may take a sphere of very great permeability as an isolated pole m. Should it seem clearer, this sphere may be thought of as the pole piece of a long magnet of infinitesimal diameter reaching to infinity, where the other pole piece forms another spherical shell. The tubes of force tend to shorten as in the electrostatic field, closing up when the sphere grows to infinite radius.

The energy per cm^3 comes out $H^2/8\pi$.

If all surrounding space is filled with a medium whose permeability is μ instead of *1*, the number of tubes for the same H is μ times as great. So, as before, we must use $\mu\psi$ and (μH) in equations (1) and (2), which, traced through, give $H(\mu H)/8\pi$ ergs per cm^3.

The first three derivations are rigorous, but awkward questions arise as to what the H and the B, in the last case, represent physically. Yet it can be made satisfactory by supposing that the long thin magnet is divided into two parts, one supplying H tubes of force, and the other supplying $(\mu - 1)$ H tubes, the former being rigidly magnetized.

From the method of this derivation it follows without additional proof that the tension along the lines of force is numerically equal to the energy density.

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