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THE TEACHING OF MATHEMATICS[1]

As mathematical pedagogy is receiving increasingly marked attention in this country, a brief account of the reforms recently proposed by Professor Felix Klein, of Germany, may be of general interest.[2]

The three types of German higher schools leading up to the university are the Gymnasium, the Real Gymnasium and the Ober Real Schule, corresponding roughly to the classical, Latin scientific and scientific courses in American high schools. Until recently, the university could only be entered through the portals of the gymnasium. This exclusive privilege might be termed the gymnasial monopoly, and even yet the gymnasium is the school of the aristocrat.

However, as a result of the Berlin Conference of 1900 and the emperor's decree of the same year, the three schools were placed on an equal footing and each per-

[1] Prepared by the author in connection with a course on the History and Teaching of Mathematics, given by Professor S. E. Slocum at the University of Cincinnati. For reports prepared by other members of the class see article soon to appear in the *Educational Review*.

[2] "Uber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen," F. Klein. "Bermerkungen im Anschluss an die Schulkonferenz von 1900," F. Klein. "Hundert Jahre mathematischer Unterricht an den höheren preussischen Schulen," F. Klein. "Uber das Lehrziel im mathematischen Unterricht der höheren Lehranstalten," E. Götting. Collected in "Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts an den Höheren Schulen," Klein u. Riecke, Teubner, 1904.

mitted to work out its ideal along its own particular line, provided that the aim is to produce cultured citizens. Graduates of the Real Gymnasium and the Ober Real Schule, however, are debarred from the study of theology, and graduates of the latter are further debarred from the profession of medicine. In addition to the exclusive rights possessed by these three schools as feeders to the university, they also have the privilege of furnishing candidates for a majority of civil service positions.

After a three years' course in a Vorschule, or equivalent work done with private tutors, the average boy enters these schools at the age of nine, and may accomplish the work in nine years, being eighteen years of age when ready to enter the university; but so extensive is their study, and so closely are they held to their work, that the graduate of these schools is considered by many to be prepared to enter the junior year of our colleges.

This thoroughness of instruction is due to the fact that in Germany teaching is a profession, and is invested with all the dignity of custom and authority. The teacher must be a university graduate and a specialist in those subjects which he expects to teach. After having completed a three to five years' university course, a year is taken for the teachers' examinations. The applicant must qualify in at least four subjects (two major and two minor), and may teach only those subjects in which he has qualified.

His examination consists of two parts: written and oral. In the first he is assigned topics upon which to prepare theses, and is given six weeks to prepare each topic. His doctor's dissertation may be offered as one of these. If the written examination is satisfactory he is orally tested to determine his readiness in commanding his specialties. If successful, he is given a certificate from the examining board, which is composed of university professors.

After securing this certificate, a year's course in theoretical pedagogy must be taken at some seminar. Then follows the Probe-jahr, or year of trial teaching under criticism. If he is finally declared proficient, his name is placed on the service list, and he ultimately secures a position, sometimes waiting six or seven years for an appointment

Adding the year of army service, the candidate is at least twenty-five years old (most of them are thirty) when placed on the list. Considering the thoroughness of preparation, and the depth of German scholarship, the statement that Germany has the best trained teachers in the world is, therefore, not surprising.

It should also be noted that all schools must come up to a certain definite standard. The government has a thorough system of inspection (as a matter of fact, too much bureaucracy), so that for a given type of school certain courses are uniform throughout.

For comparison the following outline of the course in mathematics in the Cassel Real Gymnasium is given. The period chosen corresponds most closely to that of the average American high school course.[3]

OBERSECUNDA (age, 15–16 years)

I. *Geometry and Trigonometry*, 3 hours.[4]

Plane geometry and trigonometry re-

[3] The nine years of the German high school course, beginning at the lowest, are called, respectively, Sexta, Quinta, Quarta, Untertertia, Obertertia, Untersecunda, Obersecunda, Unterprima and Oberprima. See Russell's "German Higher Schools."

[4] Since 1901, forty-two week-hours are devoted to mathematics in the Real Gymnasium, a week-hour being one hour per week throughout the year.

viewed and concluded; solid geometry; practical applications.

II. *Arithmetic and Algebra,* 2 hours.

Arithmetical and geometrical series; compound interest and annuities; quadratic equations; permutations and combinations; binomial theorem applied to positive integral exponents.

UNTERPRIMA (age, 16–17 years)

I. *Geometry and Trigonometry,* 3 hours.

Solid geometry continued; theory of plane and spherical angles; spherical trigonometry and its applications to mathematical geography; conic sections.

II. *Arithmetic and Algebra,* 2 hours.

Continued fractions and applications; arithmetical series of second order; cubic equations; problems in maxima and minima.

OBERPRIMA (age, 17–18 years)

I. *Geometry,* 3 hours.

Solid geometry reviewed and concluded; analytic geometry; problems in mathematical geography; geometrical drawing.

II. *Arithmetic and Algebra,* 2 hours.

Functions and applications to higher equations, especially those of the third degree; exponential, logarithmic, sine and cosine series; practical applications.

At the beginning of the nineteenth century those subjects whose development had been going on through the seventeenth and eighteenth centuries occupied the foreground in Germany, namely, Euclidean geometry; calculation with letters (Buchstabenrechnung); the theory of logarithms; the decimal system and the elements of analytic geometry. The elements of differential and integral calculus, although new, were also studied. The general tendency was toward the practical. Mensuration, elementary mechanics and those portions of descriptive geometry which dealt with fortifications occupied an important place. It is also noteworthy that a certain amount of mathematical knowledge was considered a prerequisite for philosophical learning, as witness the cases of Leibnitz and Kant.

Klein divides the nineteenth century into three periods. In the first period, extending from 1800 to 1870, mathematical instruction was a mixture of the pure and applied. Ideals were high, efforts were directed toward developing individual ability, and attempts were made to teach more than is now required. The candidate for the position of teacher of mathematics must be one who had gone as far as possible into the field, and was himself capable of original research. As the result we find such names as Grassmann, Kummer, Plücker, Weierstrass and Schellbach.

The second period, extending from 1870 to 1890, opened with the victory over France, and the assumption by Germany of a more important international position. This period seemed to be marked by the separation of pure, or abstract, and applied mathematics. In the schools the feeling prevailed that the development of the especially gifted pupil was not so much to be sought as that of the average pupil, and, consequently, greater interest was manifested in methods of instruction. A desire was expressed to replace the early system by a systematic graded course in mathematics, which should keep in view the ability of the constantly developing pupil. Drawings and models were demanded; problems were so stated and aids so given that pupils might see space relations, and not depend so largely upon the logic of the ancient Greeks. This was a direct result of the teachings of Pestalozzi and Herbart. In this period the teaching standard was lowered, as the teacher was only required to possess a knowledge sufficient to work out problems of moderate difficulty.

The third period, beginning with 1890, seems to be characterized by a tendency to again associate pure and applied mathematics; that is to say, the idea prevails that while a teacher should be thoroughly familiar with pure mathematics, his knowledge of its applications in the various fields should also be extensive. This is perhaps one result of the new order of things which puts the *real* schools on an equality as to privileges with the older gymnasium. There is also a tendency to allow the teacher greater freedom from the dictation of a centralized bureaucracy, and in this freedom lies an opportunity for future development.

For many decades, under the rule of the new humanism, the value of mathematical training was thought to lie in its formal discipline. Before the revival of learning it was the utilitarian factor which received emphasis, but in the last decades the majority have reached a more comprehensive view. Briefly stated, the modern view is that mathematical thought should be cherished in the schools in its fullest independence, its content being regulated in a measure by the other problems of the school; that is to say, its content should be such as to establish the liveliest possible connection with the various parts of the general culture which is typical of the school in question. Here, then, it is not a question of methods of teaching, but rather of the selection of material from the great mass furnished by elementary mathematics.

In the conference of 1900, it was agreed that each type of school should determine what form of culture its particular course should produce. It seems that the Gymnasium was asserting its claim to be considered preeminently the culture school, not hesitating to stigmatize the others as mere technical schools, while the friends of the Real schools apparently made no efforts at defense. Klein emphasized the fact that he considers the three schools of equal importance, and whatever he has to say concerns all three types.

Much of the material of instruction, although interesting in itself, lacks connection and is partially isolated. In fact, the topics seem for the most part to be the result of chance selection, and afford only a faulty and indirect preparation for a clear understanding of the mathematical element of modern culture. This element clearly rests on the idea of function and its form, both geometrical and analytical, and this idea should, therefore, be made the center of mathematical instruction. Klein's chief thesis is, in fact, that beginning with the Untersecunda and proceeding in regular, methodical steps, the geometrical concept of a function should permeate all mathematical instruction. In this is included a certain consideration of analytic geometry, and the elements of differential and integral calculus. He refers in this connection to two French publications which to a certain extent carry out his ideas.[5]

To accomplish this purpose, the graphical representation of the simplest elements, such as $y = ax + b$ and $y = 1/x$, should be begun in the Untersecunda. Trigonometry and the theory of algebraic equations furnish ample material for more complicated work, while in this connection related illustrations can be obtained from applications of mathematics, particularly from the domain of physics. Also the idea should be especially inculcated that a function can be developed empirically, perhaps by means of apparatus. In the Prima the general fundamental principles of both differential and integral calculus should be given, based upon the ideas which the pupil has acquired in the Secunda.

[5] "Notions de mathématique," Jules Tannery; "Algebra," E. Borel.

The ground to be covered depends largely upon the ideals of the school. Although the formal side must not be neglected and a thorough knowledge of processes must be obtained, the principal aim is to give a clear conception of the fundamental ideas and their meaning.

Much confusion often results from the fact that a word possesses several meanings. Thus a purist might define elementary mathematics as those parts of the subject in which the conception of a limit is avoided. The more commonly accepted definition of elementary mathematics, however, admits the idea of limits but excludes the special forms represented by the symbols dy/dx and $\int y dx$. Neither definition can be made to agree with the practise of the schools. For example, the first definition would exclude the consideration of such irrationals as $\sqrt{2}$, and π used in determining the area of a circle as the limit approached by a polygon. On the other hand, the second definition might be made to include much which does not belong in the schools, as, for example, the so-called "elementary" theory of analytic functions of the complex variable. The first definition might also be made to include much of the most difficult nature, such as advanced portions of the theory of numbers. In geometry there is also a new use of the word elementary. That portion of geometry is now styled elementary which is based on the Euclidean or ancient Greek geometry, the simplest conceptions of the newer geometry being of too severe a nature for the schools.

The only definition which will hold within the schools is a very practical one, namely, that shall be called elementary in the various branches of mathematics which can be grasped by the average pupil without extraordinary effort of long duration.

The material which constitutes elementary mathematics varies with time; that is to say, it is subject to the law of historical delay. Subjects which formerly were not considered elementary have, by improved processes of instruction, been made so, as is shown, for instance, in the geometry of the ancients. If, in consequence of the above definition, the extent of the field of elementary mathematics becomes too great and indeterminate, it comes within the province of the schools to choose those parts which best serve their purpose.

Mathematical instruction, on the level at which it is at present carried on in the upper classes of the higher schools, has existed in Germany since about the beginning of the eighteenth century. Christian Wolf, who was professor at Halle and one of the foremost schoolmen of this period, included in his list of elementary mathematics, in addition to the geometry of the ancients, a great many of what were at that time modern achievements, such as calculations with letters, negative numbers, algebraic equations and logarithms; in fact, practically everything which was known to mathematicians in 1700. It is evident that calculus was not included, for at that time the knowledge of calculus was the possession of only a few investigators of the highest type, whose efforts were not so much directed toward the clearing up of fundamental principles as toward the solution of new and difficult problems. To the layman, calculus seemed a sort of witchcraft. Cauchy's great work on differential and integral calculus appeared in 1821, but the schools had already been led into certain channels, and it was not possible to divert then toward a subject which was only in process of formation.

Moreover, it is true in general that mathematics is more susceptible than any other subject to hysteresis. A new idea finds its way into the schools through the lectures of university professors. A new generation of teachers is thus trained who give the idea shape in their class work, until finally it becomes the common possession of all. In accordance with this process of development, Klein expresses his belief that it is now time to make the fundamentals of calculus a necessary part of elementary instruction. To illustrate the historical development of the subject, he quotes the words of his teacher of mathematics, who said in the fall of 1865, "In elementary mathematics we can prove things, but in the higher mathematics it is different. They resemble a philosophical system, which we may or may not believe." It is remarkable that this idea has completely disappeared in such a short interval.

For a long time calculus was regarded with distrust, but as it received recognition in the official course of study of 1900, Klein believes that it is time to take advantage of this favorable attitude to put that which has taken centuries for preparation upon a general and recognized basis. As a matter of fact the fundamental ideas underlying the calculus are actually taught in many schools. In a few Ober-Real schools they are regularly taught as calculus, but in the majority of the schools they are given in a very roundabout manner. It amounts to this, that students are actually taught to differentiate and integrate as soon as occasion for the same arises, but the terms differential and integral are avoided.

An inspection of the text-books in current use in the higher schools shows conclusively that many of the simpler ideas of calculus are in use, but are rendered more or less difficult of comprehension by the avoidance of symbols and operations, which, if understood, would render the work comparatively easy. If the field of physics were examined, instances of this kind would be greatly multiplied, especially in the fundamentals of mechanics and electrodynamics. Evidently, then, calculus occupies a more extensive field than is commonly supposed, but it is taught unsystematically, and is merely tacked on here and there to the general content of mathematical instruction. Klein is of the opinion that instead of making instruction in calculus in those grades whose work demands its employment merely incidental, desultory and generally unsatisfactory, it should be made the central idea of all instruction, and the other ideas and work grouped around it.

At present calculus is made the beginning of higher mathematics and is accompanied by a revolution in thinking. This revolution furnishes good evidence of the aimlessness of the earlier instruction as contrasted with the ideas with which the pupil later comes in contact. Klein's suggestion aims to spare the pupil this sudden change, by gradually accustoming him to the methods of thinking which prevail in his later work.

The traditional methods of teaching will readily accommodate themselves to this new idea, and in fact will be much simplified thereby. This statement is borne out by a comparison of the cumbrous algebraic method of solving problems with the methods of calculus. On the other hand, no harm is done if certain portions of mathematics which supposedly have merely a formal training value, such as artificial equations solved by quadratic roots, and trigonometric analysis, are pushed to the rear, for the new material gives ample opportunity for formal work.

The inadequacy of the present system is clearly shown in the education of the lawyer, physician or chemist. As regards the first two, it is, of course, understood at the outset that their work in mathematics must necessarily be brief, as their major subject allows little time for it. Hence these students take up their major subject without preparation in calculus, with the result that some of the most important phases of their subject, depending upon higher mathematics, always remain obscure to them. This is true with lawyers, for instance, as regards questions of statistics, insurance, etc. With physicians the lack is felt at the very beginning of experimental physics, by reason of which instruction in the subject is necessarily placed on a much lower plane and the most important principles are only understood in a hazy way. It is still worse in chemistry, where quantitative determinations require the use of comparatively complicated formulas.

The text-books in these various subjects try to meet this situation with short prefaces on calculus, which the students are supposed to acquire in this condensed form. How then can the statement that calculus is too difficult for the higher schools be reconciled with the fact that students just released from the higher schools are expected to acquire this important subject from such condensed materials. Evidently conditions in the university emphasize the haziness of the aims of higher school mathematics.

Klein seems especially anxious to have it thoroughly understood that his plan is perfectly feasible, and comes well within the pedagogical possibilities of the case. In the first place, no more time is required than at present given to the mathematical curriculum. Moreover, he is not demanding a change in the course of study, but rather is urging that advantage be taken of the present leaning toward calculus, and carried out to its logical sequence. This, of course, can not be the work of a university professor, but must be that of the practical schoolmaster. The chief difficulty at present is that there are no text-books which fully meet the situation. Again, it is necessary to proceed with care and circumspection so as not to arouse the antagonism of the gymnasial leaders, but rather secure their friendly cooperation. There can be little opposition from the physicist if he is assured that there is no intention of invading his province, and it is pointed out to him that the pupils are being given tools for a far more complete mastery of his specialty. Neither should opposition be encountered from the representatives of the language and history departments if it is fully impressed upon their minds that the guiding principle of instruction should be the study of special subjects not as isolated from the rest of the curriculum, but with reference to the general culture which his particular type of school aims to produce.

The two main objections which are always urged when a university professor discusses educational problems of a general nature are that too little heed is given to pedagogical possibilities, and that university professors are only concerned about those pupils who will later come under their instruction. Concerning these objections, Klein answers the first by stating that he is keenly alive to the difficulty of the task of raising a large number of pupils, not especially gifted with mathematical ability, to a certain established level, and that his aim is not to raise this level, but rather to move it in what might be termed a horizontal direction.

As regards the second objection, he says that those pupils who take mathematics at the university are precisely the ones about whom he is not concerned, but that it is the future chemist, physician or lawyer whose mathematical training needs to be improved in order to bring about the best results.

Charles Ottermann
Woodward High School,
Cincinnati