This dictum from the great head of the Sibley School of Engineering is impressive and worthy of careful consideration. It fully justifies our presence here to-day for the purpose in hand and confirms the judgment of the founder of this institute and the purpose of the generous donors of the Walker Laboratory we here and now dedicate to the study of chemistry.

We may heartily join in congratulations to the administrators of the will of Mr. Van Rensselaer, who have so faithfully carried out the purpose of the founder, and to the graduates, who have so well seconded the efforts of these able and conscientious men who have brought the Rensselaer Polytechnic Institute to that great eminence in the public esteem, from which they may look backward with pride and forward in earnest and confident hope of an even brighter and more prosperous future.

WM. MCMURTRIE.

THE RATIONAL BASIS OF MATHEMATICAL PEDAGOGY[1]

THE rapid development of special methods of teaching special subjects has drawn attention of late to the hitherto neglected field of mathematical pedagogy. The fact that mathematics is the last to respond to improved pedagogical methods is due chiefly to the unusual weight of precedent which attaches to the subject. This inertia of age is, in reality, the chief difficulty to be overcome, for the great antiquity of elementary mathematics and the diversity of the sources from which it originated make it extremely difficult to harmonize the subject with the spirit of modern civilization.

Various plans have recently been proposed for adapting mathematical instruction to modern conditions, but so far they have been without results of special importance, as no general principle of mathematical pedagogy seems as yet to be recognized. Many of these plans are the results of attempts to meet local conditions and therefore have no general application. Others, however, such as the attempt to correlate mathematics and physics, are intended to stimulate interest in mathematics by establishing more points of contact between it and the other subjects of instruction, thus producing a greater organic unity in the curriculum than has hitherto existed. Such efforts are in line with the constructive and synthetic spirit which characterizes modern scientific thought, and for this reason are worthy of special consideration.

Without going into a detailed analysis of pedagogical methods, it may be well to consider briefly the fundamental principles on which general pedagogy is based, as these principles are well established and apply with peculiar force to mathematics.

The primary consideration in all branches of pedagogy is the aim of education. This has been variously defined, the Herbartian definition of it being that it is "the cultivation of virtue based on many sidedness of interest." By the cultivation of virtue in this connection must be understood the proper exercise and control of all the faculties. With this understanding the definition fits in, probably as well as any other, with modern ethical and religious ideals. The second half of the definition which bases the development of virtue on the cultivation of wide and varied interests fulfills practical requirements and at the same time affords the proper pedagogical basis for apperception. The most important feature of this definition is that it recognizes both the practical and the cultural aims of education. In other words, it implies that no training can properly be

[1] Read before American Mathematical Society, New York, December 28, 1906.

called educative in which esoteric development does not result in exoteric manifestation. This is especially important in the case of mathematical instruction which exhibits an unfortunate tendency to run to the extremes of either pure logic or empiricism.

Having defined the aim of education, the most efficient means of attaining it becomes the great problem of pedagogy. The first step towards the solution of this problem may be said to have been taken by biology in the establishment of the law of physical evolution. Following out the analogy thus suggested, modern psychology has practically solved the problem by studying the content of the child mind at different ages, thus determining the natural course of mental evolution. In this way it has been conclusively shown that mental processes follow the historical order of development, or, as Herbert Spencer expressed it, that "the genesis of knowledge in the individual follows the same course as the genesis of knowledge in the race." More recently modern psychologists have found even in the most minute activities of the child psychic atavisms as remarkable as convincing, proving conclusively that the natural course of mental evolution is but a repetition of civilization in miniature.

Since pedagogy, like engineering, is chiefly concerned with the utilization of natural forces and their direction in the proper channels, it follows that the fundamental principle underlying general pedagogy must necessarily be the historical method of presentation. In the case of mathematics this is the logical as well as the psychological sequence of development, which obviates many of the difficulties encountered in applying the historical method to other branches of instruction. The historical method thus fulfills the prime requisites for a practical working theory in being simple in application as well as powerful in results.

A notable instance of the application of the historical method to general pedagogy has already been made, and is embodied in the well-known "culture epoch" theory, originated by Pestalozzi and Herbart and elaborated by their disciples.[2] This theory consists, in brief, in applying the evolutionary idea with the utmost detail to the elementary school curriculum, with the purpose of leading the child successively through each stage of culture occupied by the race in the evolution of modern civilization. As a typical instance of the application of this method, Ziller's interpretation of the culture-epoch theory may be cited, as it is now well established in Germany on a practical footing.

In outline Ziller's method consists in arbitrarily selecting eight great historical culture epochs, corresponding as nearly as possible to the first eight years of school life. Material is then selected to embody the culture of each epoch, that chosen by Ziller being as follows: (1) epic fairy tales; (2) Robinson Crusoe; (3) history of the patriarchs; (4) history of the judges in Israel; (5) history of the kings in Israel; (6) life of Christ; (7) acts of the Apostles; (8) history of the Reformation. The subjects thus selected are known as "concentration centers" for the reason that each is used as a nucleus around which to group supplementary courses in language, science, etc. These supplementary courses are then so chosen that each group shall form a unit, representing, so far as possible, a complete stage of civilization in miniature.

The results of Ziller's method are in the main satisfactory, and at least afford a sug-

[2] See any of the numerous treatises on Herbart's educational theories, *e. g.*, "Ufer's Pedagogy of Herbart," by De Garmo; or "Introduction to Herbart's Science and Practise of Education," by Felkin.

gestive instance of the application of the historical method. It is evident, however, that Ziller's interpretation of the dual theory of the culture epochs and concentration centers is open to criticism. This is apparent in the arbitrary selection of the culture epochs, as they only partly typify the great epochs of history. Furthermore, the concentration material selected by Ziller by no means embodies the total experience of the race in any particular epoch, and for this reason is inconsistent with the principle by which it was selected. In applying the theory this weakness has also made itself felt by reason of the impossibility of reproducing historical environment, and the difficulty of adequately presenting the notable characters of ancient civilization without it. These objections have led to severe criticism of the whole culture-epoch theory, and in some cases to its entire rejection.

It should be noted, however, that the difficulties attending the culture-epoch theory are inherent in this theory and not in the historical method. In the case of mathematics the question of environment does not arise, and thus the chief difficulty is at once removed. Moreover, the nature of the subject matter in elementary mathematics is such that none of it can be omitted, thus obviating all possibility of error in the selection of proper materials.

Perhaps the most convincing proof of the applicability of the historical method to mathematics is furnished by the practical methods attained by teachers as the result of long experience. Special aptitude for teaching consists largely in the ability to assume the mental attitude of the pupil, and establish the connection between the ideas already formed and those which it is desired to communicate; or, more briefly, in the ability to stimulate apperception. By long and earnest efforts of this kind such noted teachers as De Morgan, Grube and others have arrived at methods of presentation which in the main follow the historical sequence of development, thus affording a strong inductive proof of the validity of this method. The recognition of the historical method as the universal principle underlying experience by means of which these results may be codified and extended, is, then, all that is necessary to furnish a rational basis for mathematical pedagogy.

In applying the historical method to mathematics, one of the most interesting results is the light which is thrown on the nature of the difficulties encountered in studying the subject. From the fact that mathematics has formed the basis of all civilization, and has developed independently among nations widely separated, it may be assumed that it possesses a certain universality akin to that of mind itself. This is by no means true, however, of the special branches of the subject. Thus it is by no means merely fortuitous that the Greeks excelled in geometry but produced no great algebraists, and that the reverse was the case with the Semitic races. The mathematical attainments of any nation are, in fact, an integral part of its national culture, and may, therefore, be expected to differ in direction with the latter. In so far, then, as mathematics satisfy the common needs of humanity they may justly lay claim to universality, but beyond this point are characterized by the spirit and aims of the nation which gave them birth.

It is not surprising, therefore, that those reared under modern conditions should experience difficulty in assimilating results attained hundreds or thousands of years ago and expressive of a culture entirely foreign to our own; or that they should at times fail to recognize the value of certain branches of the subject. For instance, geometry is still taught in practically the same form in which it was left by Euclid 2,200 years ago. The great Greek mathematicians, Pythagoras, Plato and Aristotle were, however, primarily philosophers, and the geometry they originated is instinct with Greek idealism. In fact the era of Greek culture may be characterized as the adolescent stage in the intellectual development of humanity. With the Greeks the worship of the ideal and the beautiful rose to the height of a religious cult, and the chief boast of the founders of geometry was that they had raised it above the common needs of humanity, and elevated it to the dignity of pure logic. This view of geometry as typical of the adolescence of the race explains why it appeals to youth and at the same time is criticized as unpractical by those of greater maturity. Euclidean geometry, however, should not be viewed from a practical standpoint only, for since the full power of maturity is only attained by the proper unfolding of the preceding stages of childhood and youth, geometry is an important factor in development, and should not give place to more utilitarian subjects until it has fully served its purpose as a mental stimulus.

As geometry is a characteristic expression of Greek culture, so algebra sprung from, and fitly symbolizes, the mystic spirit of the Hindoos and their Aryan conquerors. For the modern youth, therefore, the difficulties met with in geometry are by no means so great as those encountered in algebra, for the Greek spirit in its two chief features of freedom and individuality has much in common with our own, whereas that of the Hindoos is its exact antithesis. Fettered by the bonds of caste, the Hindoo spirit could not attain objective realization and became lost in a maze of abstraction; the highest good becoming a mere negation of existence both physical and intellectual. Moreover, the divinity ascribed to the Brahmin caste resulted in the degradation of religion, and the absorption of the spiritual in the merely physical. Thus the morality involved in respect for life and its Creator was lost, and the ideal of virtue was abstraction from all activity. In short, concrete reality gave way to abstraction, imagination became dominant, and spirit was characterized by the fanciful imaginings of dreams. The difficulties met with in algebra are therefore inherent in the thought processes involved and can only be lessened by establishing relations with more familiar ideas by the frequent introduction of concrete numerical illustrations.

Besides explaining the nature of the difficulties encountered, the historical method also furnishes a means of estimating their relative magnitude. In other words, the historical method affords a criterion for making a quantitative as well as a qualitative estimate of the intellectual content of the subjects considered. Thus the long period of time occupied by the Egyptians in reducing fractions to a working basis is significant as being the prototype of the serious difficulty experienced by the modern youth in attaining an equal proficiency in the subject. The pause which frequently intervenes between two successive stages of development is also significant, and is analogous to that which occurs at intervals in the growth of the child. As in the case of physical growth, so here, the pause marks a drop in potential due to accelerated development, and its length indicates the importance of the next successive advance. It is in fact a sort of hysteresis due to the mental inertia of the race. Political and religious conditions are not sufficient to account for such halts in progress. Although social conditions must be recognized as powerful factors in aiding or arresting development, yet from the standpoint of universal history such external relations can not be considered as conditioning the evolution of spirit, but rather as reflecting its trend.

To illustrate the meaning of a pause such as mentioned, the hiatus of sixteen centuries which intervened between the statics of Archimedes and the dynamics of Stevinus and Galileo may be cited. The difficulty experienced by students in passing from statics to kinetics has frequently been remarked by teachers, and has led to a revision of instruction in mechanics by beginning the subject with kinematics and treating statics as a special case of dynamics. This order of development might, however, have been inferred from the historic relation of the subjects, for the history of mechanics shows that the statics of Archimedes consisted in little more than the law of the lever, and that no advance was made until the subject was approached from the standpoint of motion. In fact such an elementary principle of statics as the parallelogram of forces was not proved or even commonly accepted until after the enunciation of Newton's laws of motion. In this case, then, the pause emphasizes the degree of attainment essential to a proper understanding of the laws of motion, and also the necessity of approaching the subject from the proper direction, both of which are of the greatest pedagogical importance.

The long interval of time, approximating 4,000 years, which was spent by the ancients in acquiring the fundamental ideas of number is another instance in point, and indicates the necessity of thoroughness in the first stages of instruction. Here again theory has been anticipated by experience in the method proposed by Grube, which consists in spending the entire two first years of mathematical instruction in exhaustive number analysis. There is no doubt but that under the present forcing system too little time is devoted to this basic work, the result being that ability to make numerical calculations with ease and facility is the exception rather than the rule. This also explains the reason for the unfavorable comparison sometimes drawn between our modern schools with their multiplicity of subjects and too often superficial treatment, and the old red schoolhouse of the last generation, where instruction was limited to the three R's, but where each was taught with such thoroughness as to leave a permanent impress on the character of the scholar.

The movement recently inaugurated in Germany and England with a view to revising the present instruction in mathematics indicates the lack of harmony between ancient and modern civilization. A characteristic expression of this dissatisfaction with existing methods of instruction is found in the so-called "Perry movement" and its rapid spread throughout England and America. The chief feature of the modification proposed by Professor Perry is the laboratory method of instruction, which may be characterized as an attempt to visualize mathematics, at the same time making it utilitarian as well as concrete. It is, therefore, a reversion to the basic needs of humanity and the means which were used for supplying them. Thus arithmetic originated with the Phœnicians and Chaldeans to supply their commercial needs, while even with the Greeks the beginnings of geometry may be traced to the attempt to solve certain practical problems in mensuration. It is, in fact, a general truth that the chief stimulus to the development of mathematics has always been found in the attempt to explain natural phenomena, and make them subservient to the physical needs of humanity. The laboratory method, then, may be used as a basis for the inductive development of mathematics, and if used for this purpose in the lower grades of instruction will prove a valuable adjunct to the methods ordinarily followed. So far from being unique in its inception and aims, however, it is merely a corollary of the historical method, and can only be used to advantage when it is recognized as such.

Another corollary of the historical method is what is known as the "spiral method" of instruction. This consists in taking the pupil several times over the same ground, but each time reaching a higher level and attaining a more general point of view. The method is founded mainly on experience, but its theoretical basis is evidently historical.

The specific application of the historical method to mathematical pedagogy consists primarily in obtaining the proper historical perspective. From this aspect its principal use is in arranging the details of a curriculum, and a few suggestions follow relative to its application for this purpose.

Perhaps the most obvious suggestion is that subjects which developed simultaneously should form parallel courses instead of being taught serially, as is now common in all mathematical instruction. For instance, algebra and geometry originated simultaneously and served as a mutual stimulus to growth and development. It is evident, therefore, that it is possible to teach these subjects in the same academic grade, and that they can undoubtedly be made mutually helpful by so doing. This opinion is verified by the fact that this method has been used for some time in the higher schools of Prussia with results which indicate a decided advantage for such correlation of subjects.[3]

Following out the historical idea, the curriculum should be based on a thorough grounding in the principles of number, the amount of time devoted to the several subjects being proportionate to their relative difficulty as indicated by their historical rate of development. This should be followed by a course in elementary algebra, taught as a generalization of arithmetical ideas, and accompanied by a parallel course in elementary geometry. The course in elementary algebra would naturally consist in a logical development of the six fundamental processes, including logarithms. At present the latter usually follows quadratic equations and the binomial theorem, whereas historically it precedes both. The natural sequence is, in fact, to teach multiplication as an abbreviation of addition, thus leading to the theory of exponents, and then passing to logarithms as an abbreviation of multiplication. Historically the subject of logarithms arose in this connection, having been invented by Napier about 1614 for the purpose of facilitating the long numerical calculations fashionable in his day.

Conforming to the natural lines of demarkation, these elementary courses would be succeeded by advanced courses in algebra and solid geometry, the former beginning with simple equations and emphasizing chiefly the theory of equations. At present the natural sequence is not followed in teaching algebra, at least three subjects, namely, proportion, logarithms and series being out of proper historical perspective. Proportion, or the old-fashioned "rule of three," was developed by the Hindoos for the solution of numerical equations by the "rule of false assumption," and as it is now obsolete for this purpose, does not properly belong in algebra, and should be reserved for arithmetic and geometry, where it properly has a place. The proper setting for logarithms has already been mentioned. As

[3] J. W. A. Young, "The Teaching of Mathematics in the Higher Schools of Prussia."

regards series, great difficulty is usually experienced in grasping the idea of convergency and divergency at the point where it ordinarily occurs in current text-books, the reason being that it involves the idea of functionality which is of comparatively recent development. Euler first noticed in 1748 that convergency of series was necessary for computation and partly developed the idea of functionality, but the subject did not receive adequate consideration until demanded by the development of the calculus.

Two other points may be noted in connection with the teaching of algebra. The first is that the graphical method of representing an equation was originated by Descartes, who was also one of the foremost in developing the theory of equations. The inference is that graphs may be advantageously used to illustrate the theory of equations, and will also serve as a natural transition to analytic geometry. In this way the historical method meets the objection sometimes raised to our present method of instruction as being conducted in "water-tight compartments."

The second point is that the examples used to illustrate principles should be so chosen as to stimulate interest, and in order to accomplish this purpose must reflect modern life and local conditions. That this principle of selection was formerly recognized, or at least followed, is shown by some of the time-honored problems which unfortunately still survive. Thus the length of time required to fill or empty a vessel by several pipes had a practical bearing when time was measured by a clepsydra, while such problems as that of the couriers, and the length of time required by several men to complete a piece of work, were exceeding useful and interesting a century ago, but, now, have no vital interest except perhaps for the historian. The retention of such problems in modern elementary texts is evidence that the spirit of scholasticism is not yet extinct, and largely accounts for the growing chasm between mathematics and the humanities. Modern life in its growing complexity is teeming with possibilities of mathematical illustration, constantly presenting new problems far greater in cultural value and more wide-reaching in practical significance than any that have yet appeared. To revitalize instruction in elementary mathematics the pupil must be taught to recognize the true significance of mathematics, as the most powerful instrument yet devised by man for ameliorating his physical condition and reconciling cause with effect. Philosophy can never be the proper food for childhood and youth; in elementary instruction the essential feature is that it shall be instinct with life and experience.

It is beyond the scope of this article to do more than point out the chief features of the historical method and its application to mathematics. In mathematical pedagogy the present problem is one of adjustment to modern conditions. This demands for its general solution a wide outlook over the history of the past as well as an intimate knowledge of the needs of the present. The routine of teaching too often proves fatal to this breadth of view, leading the teacher into the error of measuring his success by the facility acquired by his pupils in the subject taught. The true criterion of success in instruction is whether or not it leads the pupil to his highest individual development, refining his spirit and enlarging his field of usefulness. Like other fine arts, teaching can never be made amenable to fixed rules and rigid methods. There are, however, certain general underlying principles which distinguish the art from pure caprice, and of these the historical method of presentation is fundamental.

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SCIENTIFIC BOOKS

A Student's Manual of a Laboratory Course in Physical Measurements. By Professor W. C. SABINE. Ginn & Co. 1906. 8vo, pp. 97.

A Text-book of Practical Physics. By W. WATSON. London, Longmans, Green & Co. 1906. 8vo, pp. 626.

Elementary physical laboratory work in American universities has to fulfill two requirements, namely, to complement the first lecture course in general physics and to teach accuracy of observation. While it is desirable that the number of experiments to be performed should not be too limited, the characteristic value of physics as a culture study lies in the training of accuracy of expression and observation. In order to enable the student to perform a sufficiently large number of experiments—which is unfortunately often made the test of ability—and to give him the necessary training in accuracy, it has become the custom to describe only those exercises which he is expected to perform, and avoid a possible "waste of time" by rather minute descriptions of apparatus.

The selection of a few out of a large number of instructive experiments is always a difficult task and will lead to a different choice, according to the tastes of the author and the equipment of the school in which the book is to be used.

Sabine's well-known manual which has now appeared in its second edition shows this elasticity of selection in the omission of many exercises found in the former edition and the introduction of several new ones. Their number has been reduced from about seventy to thirty. Mechanics has practically remained unchanged. In sound a qualitative experiment, "Quality by the manometric flame," has been added. All the former experiments in heat have been omitted and a single one substituted for them, namely, "the determination of the mechanical equivalent of heat." Without questioning the great importance of this exercise it seems to the reviewer that some of the discarded experiments, as "specific heat, heat of fusion, or expansion" are better adapted, at least for an elementary course which is expected to teach only the rudiments of physical manipulations. In light also important changes have been made. "Equivalent focal length of compound lenses" takes the place of several exercises on radii of curvature and focal length of mirrors and lenses. "Wave-length of light by Newton's rings and the diffraction grating," also "Rotation of polarized light" are new. In the electrical part a good descriptive chapter on galvanometers adds much to the value of the book. The work with cells (internal resistance, different arrangement of cells, etc.) has been considerably condensed and an experiment with the dynamo added.

On the whole the changes made for the new edition are good; each exercise illustrates an important principle and a repetition of the same in other parts of the book has been carefully avoided. The instructions given for each experiment are more specific than in the first edition, but this has not been carried so far as to prevent a certain independence of the student and a possible variation of the apparatus used in the course.

Watson's "text-book" is of an entirely different character. It is more of the nature of Kohlrausch's "Leitfaden" and contains nearly 200 experiments. An introduction of forty pages treats of general methods used in the reduction and discussion of the results of physical measurements, and an appendix of twenty pages contains short practical information as to glass blowing, work with fused quartz, silvering glass, mounting of cross wires in telescopes and microscopes and the use of manganin wire for the construction of standard coils.

The book is intended for students who "have already spent a little time in the laboratory," and for such it is an excellent refer-