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CONTENTS:

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*ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATIONS TO SOME OTHER SCIENCES.**

IT is one of the objects of a congress such as this which now brings us together, to show the bonds between the diverse parts of science taken in its most extended acceptation. So the organizers of this meeting have insisted that the relations between different sections should be put in evidence.

To undertake a study of this sort, somewhat indeterminate in character, it is necessary to forget that all is in all; in what concerns algebra and analysis, a Pythagorean would be dismayed at the extent of his task, remembering the celebrated formula of the school: 'Things are numbers.' From this point of view my subject would be inexhaustible.

But I, for the best of reasons, will make no such pretensions.

In casting merely a glance over the development of our science through the ages, and particularly in the last century, I hope to be able to characterize sufficiently the rôle of mathematical analysis in its relations to certain other sciences.

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I.

It would appear natural to commence by speaking of the concept itself of whole number; but this subject is not alone of logical order, it is also of order historic and psychologic, and would draw us away into too many discussions.

Since the concept of number has been sifted, in it have been found unfathomable depths; thus, it is a question still pending to know, between the two forms, the cardinal number and the ordinal number, under which the idea of number presents itself, which of the two is anterior to the other, that is to say, whether the idea of number properly so called is anterior to that of order, or if it is the inverse.

It seems that the geometer-logician neglects too much in these questions psychology and the lessons uncivilized races give us; it would seem to result from these studies that the priority is with the cardinal number.

It may also be there is no general response to the question, the response varying according to races and according to mentalities.

I have sometimes thought, on this subject, of the distinction between auditives and visuals, auditives favoring the ordinal theory, visuals the cardinal.

But I will not linger on this ground full of snares; I fear that our modern school of logicians with difficulty comes to agreement with the ethnologists and biologists; these latter in questions of origin are always dominated by the evolution idea, and, for more than one of them, logic is only the résumé of ancestral experience. Mathematicians are even reproached with postulating in principle that there is a human mind in some way exterior to things, and that it has its logic. We must, however, submit to this, on pain of constructing nothing. We need this point of departure, and certainly, supposing it to have evolved during the course of prehistoric time, this logic of the human mind was perfectly fixed at the time of the oldest geometric schools, those of Greece; their works appear to have been its first code, as is expressed by the story of Plato writing over the door of his school 'Let no one not a geometer enter here.'

Long before the bizarre word algebra was derived from the Arabic, expressing, it would seem, the operation by which equalities are reduced to a certain canonic form, the Greeks had made algebra without knowing it; relations more intimate could not be imagined than those binding together their algebra and their geometry, or rather, one would be embarrassed to classify, if there were occasion, their geometric algebra, in which they reason not on numbers but on magnitudes.

Among the Greeks also we find a geometric arithmetic, and one of the most interesting phases of its development is the conflict which, among the Pythagoreans, arose in this subject between number and magnitude, apropos of irrationals.

Though the Greeks cultivated the abstract study of numbers, called by them arithmetic, their speculative spirit showed little taste for practical calculation, which they called logistic.

In remote antiquity, the Egyptians and the Chaldeans, and later the Hindus and the Arabs, carried far the science of calculation.

They were led on by practical needs; logistic preceded arithmetic, as land-surveying and geodesy opened the way to geometry; in the same way, trigonometry developed under the influence of the increasing needs of astronomy.

The history of science at its beginnings shows a close relation between pure and applied mathematics; this we will meet again constantly in the course of this study.

We have remained up to this point in the domain which ordinary language calls elementary algebra and arithmetic.

In fact, from the time that the incommensurability of certain magnitudes had been recognized, the infinite had made its appearance, and, from the time of the sophisms of Zeno on the impossibility of motion, the summation of geometric progressions must have been considered.

The procedures of exhaustion which are found in Eudoxus and in Euclid appertain already to the integral calculus, and Archimedes calculates definite integrals.

Mechanics also appeared in his treatise on the quadrature of the parabola, since he first finds the surface of the segment bounded by an arc of a parabola and its chord with the help of the theorem of moments; this is the first example of the relations between mechanics and analysis which since have not ceased developing.

The infinitesimal method of the Greek geometers for the measure of volumes raised questions whose interest is even today not exhausted.

In plane geometry, two polygons of the same area are either equivalent or equivalent-by-completion, that is to say, can be decomposed into a finite number of triangles congruent in pairs, or may be regarded as differences of polygons susceptible of such a partition.

It is not the same for the geometry of space, and we have lately learned that stereometry can not, like planimetry, get on without recourse to procedures of exhaustion or of limit, which require the axiom of continuity or the Archimedes assumption.

Without insisting further, this hasty glance at antiquity shows how completely then were amalgamated algebra, arithmetic, geometry and the first endeavors at integral calculus and mechanics, to the point of its being impossible to recall separately their history.

In the middle ages and the renaissance, the geometric algebra of the ancients separated from geometry. Little by little algebra properly so called arrived at independence, with its symbolism and its notation more and more perfected; thus was created this language so admirably clear, which brings about for thought a veritable economy and renders further progress possible.

This is also the moment when distinct divisions are organized.

Trigonometry which, in antiquity, had been only an auxiliary of astronomy is developed independently; toward the same time the logarithm appears, and essential elements are thus put in evidence.

II.

In the seventeenth century, the analytic geometry of Descartes, distinct from what I have just called the geometric algebra of the Greeks by the general and systematic ideas which are at its base, and the newborn dynamic were the origin of the greatest progress of analysis.

When Galileo, starting from the hypothesis that the velocity of heavy bodies in their fall is proportional to the time, from this deduced the law of the distances passed over, to afterward verify it by experiment, he took up again the road upon which Archimedes had formerly entered and on which would follow after him Cavalieri, Fermat and others still, even to Newton and Leibnitz. The integral calculus of the Greek geometers was born again in the kinematic of the great Florentine physicist.

As to the calculus of derivatives or of differentials, it was founded with precision apropos of the drawing of tangents.

In reality, the origin of the notion of derivative is in the confused sense of the mobility of things and of the rapidity more or less great with which phenomena happen; this is well expressed by the words *fluents* and *fluxions,* which Newton used, and which one might suppose borrowed from old Heraclitus.

The points of view taken by the founders of the science of motion, Galileo, Huygens and Newton, had an enormous influence on the orientation of mathematical analysis.

It was with Galileo an intuition of genius to discover that, in natural phenomena, the determining circumstances of the motion produce accelerations: this must have conducted to the statement of the principle that the rapidity with which the dynamic state of a system changes depends in a determinate manner on its static state alone. In a more general way we reach the postulate that the infinitesimal changes, of whatever nature they may be, occurring in a system of bodies, depend uniquely on the actual state of this system.

In what degree are the exceptions apparent or real? This is a question which was raised only later and which I put aside for the moment.

From the principles enunciated becomes clear a point of capital importance for the analyst: Phenomena are ruled by differential equations which can be formed when observation and experiment have made known for each category of phenomena certain physical laws.

We understand the unlimited hopes conceived from these results. As Bertrand says in the preface of his treatise, 'the early successes were at first such that one might suppose all the difficulties of science surmounted in advance, and believe that the geometers, without being longer distracted by the elaboration of pure mathematics, could turn their meditations exclusively toward the study of the natural laws.'

This was to admit gratuitously that the problems of analysis, to which one was led, would not present very grave difficulties.

Despite the disillusions the future was to bring, this capital point remained, that the problems had taken a precise form, and that a classification could be established in the difficulties to be surmounted.

There was, therefore, an immense advance, one of the greatest ever made by the human mind. We understand also why the theory of differential equations acquired a considerable importance.

I have anticipated somewhat, in presenting things under a form so analytic. Geometry was intermingled in all this progress. Huygens, for example, followed always by preference the ancients, and his 'Horologium oscillatorium' rests at the same time on infinitesimal geometry and mechanics; in the same way, in the 'Principia' of Newton, the methods followed are synthetic.

It is above all with Leibnitz that science takes the paths which were to lead to what we call mathematical analysis; it is he who, for the first time, in the latter years of the seventeenth century, pronounces the word function.

By his systematic spirit, by the numerous problems he treated, even as his disciples James and John Bernoulli, he established in a final way the power of the doctrines to the edification of which had successively contributed a long series of thinkers from the distant times of Eudoxus and of Archimedes.

The eighteenth century showed the extreme fecundity of the new methods. That was a strange time, the era of mathematical duels where geometers hurled defiance, combats not always without acrimony, when Leibnitzians and Newtonians encountered in the lists.

From the purely analytic point of view, the classification and study of simple functions is particularly interesting; the function idea, on which analysis rests, is thus developed little by little.

The celebrated works of Euler hold then a considerable place. However, the numerous problems which present themselves to the mathematicians leave no time for a scrutiny of principles; the foundations themselves of the doctrine are elucidated slowly, and the *mot* attributed to d'Alembert 'allez en avant et la foi vous viendra' is very characteristic of this epoch.

Of all the problems started at the end of the seventeenth century or during the first half of the eighteenth, it will suffice for me to recall those isoperimetric problems which gave birth to the calculus of variations.

I prefer to insist on the interpenetration still more intimate between analysis and mechanics when, after the inductive period of the first age of dynamics, the deductive period was reached where one strove to give a final form to the principles. The mathematical and formal development played then the essential rôle, and the analytic language was indispensable to the greatest extension of these principles.

There are moments in the history of the sciences and, perhaps, of society, when the spirit is sustained and carried forward by the words and the symbols it has created, and when generalizations present themselves with the least effort. Such was particularly the rôle of analysis in the formal development of mechanics.

Allow me a remark just here. It is often said an equation contains only what one has put in it. It is easy to answer, first, that the new form under which one finds the things constitutes often of itself an important discovery.

But sometimes there is more; analysis, by the simple play of its symbols, may suggest generalizations far surpassing the primitive outline. Is it not so with the principle of virtual velocities, of which the first idea comes from the simplest mechanisms; the analytic form which translates it will suggest extensions leading far from the point of departure.

In the same sense, it is not just to say analysis has created nothing, since these more general conceptions are its work. Still another example is furnished us by Lagrange's system of equations; here calculus transformations have given the type of differential equations to which one tends to carry back to-day the notion of mechanical explanation.

There are in science few examples comparable to this, of the importance of the form of an analytic relation and of the power of generalization of which it may be capable.

It is very clear that, in each case, the generalizations suggested should be made precise by an appeal to observation and experiment, then it is still the calculus which searches out distant consequences for checks, but this is an order of ideas which I need not broach here.

Under the impulse of the problems set by geometry, mechanics and physics, we see develop or take birth almost all the great divisions of analysis. First were met equations with a single independent variable. Soon appear partial differential equations, with vibrating cords, the mechanics of fluids and the infinitesimal geometry of surfaces.

This was a wholly new analytic world; the origin itself of the problems treated was an aid which from the first steps permits no wandering, and in the hands of Monge geometry rendered useful services to the new-born theories.

But of all the applications of analysis, none had then more renown than the problems of celestial mechanics set by the knowledge of the law of gravitation and to which the greatest geometers gave their names.

Theory never had a more beautiful triumph; perhaps one might add that it was too complete, because it was at this moment above all that were conceived for natural philosophy the hopes at least premature of which I spoke above.

In all this period, especially in the second half of the eighteenth century, what strikes us with admiration and is also somewhat confusing, is the extreme importance of the applications realized, while the pure theory appeared still so ill assured. One perceives it when certain questions are raised like the degree of arbitrariness in the integral of vibrating cords, which gives place to an interminable and inconclusive discussion.

Lagrange appreciated these insufficiencies when he published his theory of analytic functions where he strove to give a precise foundation to analysis.

One can not too much admire the marvelous presentiment he had of the rôle which were to play the functions we call, with him, analytic; but we may confess that we stand astonished before the demonstration he believed to have given of the possibilty of the development of a function in Taylor's series.

The exigencies in questions of pure analysis were less at this epoch. Confiding in intuition, one was content with certain probabilities and agreed implicitly about certain hypotheses that it seemed useless to formulate in an explicit way; in reality, one had confidence in the ideas which so many times had shown themselves fecund, which is very nearly the *mot* of d'Alembert.

The demand for rigor in mathematics has had its successive approximations, and in this regard our sciences have not the absolute character so many people attribute to them.

III.

We have now reached the first years of the nineteenth century. As we have explained, the great majority of the analytic researches had, in the eighteenth century, for occasion a problem of geometry, and especially of mechanics and of physics, and we have scarcely found the logical and esthetic preoccupations which are to give a physiognomy so different to so many mathematical works, above all in the latter two thirds of the nineteenth century.

Not to anticipate, however, after so many examples of the influences of physics on the developments of analysis, we meet still a new one, and one of the most memorable, in Fourier's theory of heat. He commences by forming the partial differential equations which govern temperature.

What are for a partial differential equation the conditions at the limits permitting the determination of a solution?

For Fourier, the conditions are suggested by the physical problem and the methods that he followed have served as models to the physicist-geometers of the first half of the last century.

One of these consists in forming a series with certain simple solutions. Fourier thus obtained the first types of developments more general than the trigonometric developments, as in the problem of the cooling of a sphere, where he applies his theory to the terrestrial globe, and investigates the law which governs the variations of temperature in the ground, trying to go even as far as numerical applications.

In the face of so many beautiful results, we understand the enthusiasm of Fourier which scintillates from every line of his preliminary discourse. Speaking of mathematical analysis, he says, "there could not be a language more universal, more simple, more exempt from errors and from obscurities, that is to say, more worthy to express the invariable relations of natural things. Considered under this point of view, it is as extended as nature herself; it defines all sensible relations, measures times, spaces, forces, temperatures. This difficult science forms slowly, but it retains all the principles once acquired. It grows and strengthens without cease in the midst of so many errors of the human mind.''

The elegy is magnificent, but permeating it we see the tendency which makes all analysis uniquely an auxiliary, however incomparable, of the natural sciences, a tendency, in conformity, as we have seen, with the development of science during the preceding two centuries; but we reach just here an epoch where new tendencies appear.

Poisson having in a report on the Fundamenta recalled the reproach made by Fourier to Abel and Jacobi of not having occupied themselves preferably with the movement of heat, Jacobi wrote to Legendre: 'It is true that Monsieur Fourier held the view that the principal aim of mathematics was public utility, and the explanation of natural phenomena; but a philosopher such as he should have known that the unique aim of science is the honor of the human spirit, and that from this point of view, a question about numbers is as important as a question about the system of the world.' This was without doubt also the opinion of the grand geometer of Goettingen, who called mathematics the queen of the sciences, and arithmetic the queen of mathematics.

It would be ridiculous to oppose one to the other these two tendencies; the harmony of our science is in their synthesis.

The time was about to arrive when one would feel the need of inspecting the foundations of the edifice, and of making the inventory of accumulated wealth, using more of the critical spirit. Mathematical thought was about to gather more force by retiring into itself; the problems were exhausted for a time, and it is not well for all seekers to stay on the same road. Moreover, difficulties and paradoxes remaining unexplained made necessary the progress of pure theory.

The path on which this should move was traced in its large outlines, and there it could move with independence without necessarily losing contact with the problems set by geometry, mechanics and physics.

At the same time more interest was to attach to the philosophic and artistic side of mathematics, confiding in a sort of preestablished harmony between our logical and esthetic satisfactions and the necessities of future applications.

Let us recall rapidly certain points in the history of the revision of principles where Gauss, Cauchy and Abel likewise were laborers of the first hour. Precise definitions of continuous functions, and their most immediate properties, simple rules on the convergence of series, were formulated; and soon was established under very general conditions, the possibility of trigonometric developments, legitimatizing thus the boldness of Fourier.

Certain geometric intuitions relative to areas and to arcs give place to rigorous demonstration. The geometers of the eighteenth century had necessarily sought to give account of the degree of the generality of the solution of ordinary differential equations. Their likeness to equations of finite differences led easily to the result; but the demonstration so conducted must not be pressed very close.

Lagrange, in his lessons on the calculus of functions had introduced greater precision, and starting from Taylor's series, he saw that the equation of order m leaves indeterminate the function, and its $m-1$ first derivatives for the initial value of the variable; we are not surprised that Lagrange did not set himself the question of convergence.

In twenty or thirty years the exigencies in the rigor of proofs had grown. One knew that the two preceding modes of demonstration are susceptible of all the precision necessary.

For the first, there was need of no new principle; for the second it was necessary that the theory should develop in a new way. Up to this point, the functions and the variables had remained real. The consideration of complex variables comes to extend the field of analysis. The functions of a complex variable with unique derivative are necessarily developable in Taylor's series; we come back thus to the mode of development of which the author of the theory of analytic functions had understood the interest, but of which the importance could not be put fully in evidence in limiting oneself to real variables. They also owe the grand rôle that they have not ceased to play to the facility with which we can manage them, and to their convenience in calculation.

The general theorems of the theory of analytic functions permitted to reply with precision to questions remaining up to that time undecided, such as the degree of generality of the integrals of differential equations. It became possible to push even to the end the demonstration sketched by Lagrange for an ordinary differential equation. For a partial differential equation or a system of such equations, precise theorems were established. It is not that on this latter point the results obtained, however important they may be, resolve completely the diverse questions that may be set; because in mathematical physics, and often in geometry, the conditions at the limits are susceptible of forms so varied that the problem called Cauchy's appears often under very severe form. I will shortly return to this capital point.

IV.

Without restricting ourselves to the historic order, we will follow the development of mathematical physics during the last century, in so far as it interests analysis.

The problems of calorific equilibrium lead to the equation already encountered by Laplace in the study of attraction. Few equations have been the object of so many works as this celebrated equation. The conditions at the limits may be of divers forms. The simplest case is that of the calorific equilibrium of a body of which we maintain the elements of the surface at given temperatures; from the physical point of view, it may be regarded as evident that the temperature, continuous within the interior since no source of heat is there, is determined when it is given at the surface.

A more general case is that where, the state remaining permanent, there might be radiation toward the outside with an emissive power varying on the surface in accordance with a given law; in particular the temperature may be given on one portion, while there is radiation on another portion.

These questions, which are not yet resolved in their greatest generality, have greatly contributed to the orientation of the theory of partial differential equations. They have called attention to types of determinations of integrals, which would not have presented themselves in remaining at a point of view purely abstract.

Laplace's equation had been met already in hydrodynamics and in the study of attraction inversely as the square of the distance. This latter theory has led to putting in evidence the most essential elements such as the potentials of simple strata and of double strata. Analytic combinations of the highest importance were there met, which since have been notably generalized, such as Green's formula.

The fundamental problems of static electricity belong to the same order of ideas, and that was surely a beautiful triumph for theory, the discovery of the celebrated theorem on electric phenomena in the interior of hollow conductors, which later Faraday rediscovered experimentally, without having known of Green's memoir.

All this magnificent ensemble has remained the type of the theories already old of mathematical physics, which seem to us almost to have attained perfection, and which exercise still so happy an influence on the progress of pure analysis in suggesting to it the most beautiful problems. The theory of functions offers us another memorable affiliation.

There the analytic transformations which come into play are not distinct from those we have met in the permanent movement of heat. Certain fundamental problems of the theory of functions of a complex variable lost then their abstract enunciation to take a physical form, such as that of the distribution of temperature on a closed surface of any connection and not radiating, in calorific equilibrium with two sources of heat which necessarily correspond to flows equal and of contrary signs. Transposing, we face a question relative to Abelian integrals of the third species in the theory of algebraic curves.

The examples which precede, where we have envisaged only the equations of heat and of attraction, show that the influence of physical theories has not been exercised only on the general nature of the problems to be solved, but even in the details of the analytic transformations. Thus is currently designated in recent memoirs on partial differential equations under the name of Green's formula, a formula inspired by the primitive formula of the English physicist. The theory of dynamic electricity and that of magnetism, with Ampère and Gauss, have been the origin of important progress; the study of curvilinear integrals and that of the integrals of surfaces have taken thence all their developments, and formulas, such as that of Stokes which might also be called Ampère's formula, have appeared for the first time in memoirs on physics. The equations of the propagation of electricity, to which are attached the names of Ohm and Kirchoff, while presenting a great analogy with those of heat, offer often conditions at the limits a little different; we know all that telegraphy by cables owes to the profound discussion of a Fourier's equation carried over into electricity.

The equations long ago written of hydrodynamics, the equations of the theory of electricity, those of Maxwell and of Hertz in electromagnetism have offered problems analogous to those recalled above, but under conditions still more varied. Many unsurmounted difficulties are there met with; but how many beautiful results we owe to the study of particular cases, whose number one would wish to see increase. To be noted also as interesting at once to analysis and physics are the profound differences which the propagation may present according to the phenomena studied. With equations such as those of sound, we have propagation by waves; with the equation of heat, each variation is felt instantly at every distance, but very little at a very great distance, and we can not then speak of velocity of propagation.

In other cases of which Kirchoff's equation relative to the propagation of electricity with induction and capacity offers the simplest type, there is a wave front with a velocity determined but with a remainder behind which does not vanish.

These diverse circumstances reveal very different properties of integrals; their study has been delved into only in a few particular cases, and it raises questions into which enter the most profound notions of modern analysis.

V.

I will enter into certain analytic details especially interesting for mathematical physics.

The question of the generality of the solution of a partial differential equation has presented some apparent paradoxes. For the same equation, the number of arbitrary functions figuring in the general integral was not always the same, following the form of the integral envisaged. Thus Fourier, studying the equation of heat in an indefinite medium, considers as evident that a solution will be determined if its value for $t=0$ is given, that is to say one arbitrary function of the three coordinates x, y, z; from the point of view of Cauchy, we may consider, on the contrary, that in the general solution there are two arbitrary functions of the three variables. In reality, the question, as it has long been stated, has not a precise signification.

In the first place, when it is a question only of analytic functions, any finite number of functions of any number of independent variables presents, from the arithmetical point of view, no greater generality than a single function of a single variable, since in the one case and in the other the ensemble of coefficients of the development forms an enumerable series. But there is something more. In reality, beyond the conditions which are translated by given functions, an integral is subjected to conditions of continuity, or is to become infinite in a determined manner for certain elements; one may so be led to regard as equivalent to an arbitrary function the condition of continuity in a given space, and then we clearly see how badly formulated is the question of giving the number of the arbitrary functions. It is at times a delicate matter to demonstrate that conditions determine in a unique manner a solution, when we do not wish to be contented with probabilities; it is then necessary to make precise the manner in which the function and certain of its derivatives conduct themselves.

Thus in Fourier's problem relative to an indefinite medium certain hypotheses must be made about the function and its first derivatives at infinity, if we wish to establish that the solution is unique.

Formulas analogous to Green's render great services, but the demonstrations one deduces from them are not always entirely rigorous, implicitly supposing fulfilled for the limits conditions which are not, *a priori* at least, necessary. This is, after so many others, a new example of the evolution of exigencies in the rigor of proofs.

We remark, moreover, that the new study, rendered necessary, has often led to a better account of the nature of integrals.

True rigor is fecund, thus distinguishing itself from another purely formal and tedious, which spreads a shadow over the problems it touches.

The difficulties in the demonstration of the unity of a solution may be very different according as it is question of equations of which all the integrals are or are not analytic. This is an important point, and shows that even when we wish to put them aside, it is necessary sometimes to consider non-analytic functions.

Thus we can not affirm that Cauchy's problem determines in a unique manner, one solution, the data of the problem being general, that is to say not being characteristic.

This is surely the case, if we envisage only analytic integrals, but with non-analytic integrals, there may be contacts of order infinite. And theory here does not outstrip applications; on the contrary, as the following example shows:

Does the celebrated theorem of Lagrange

on the potentials of velocity in a perfect fluid hold good in a viscid fluid? Examples have been given where the coordinates of different points of a viscous fluid starting from rest are not expressible as analytic functions of the time starting from the initial instant of the motion, and where the nul rotations as well as all their derivatives with respect to the time at this instant are, however, not identically nul; Lagrange's theorem, therefore, does not hold true.

These considerations sufficiently show the interest it may have to be assured that all the integrals of a system of partial differential equations continuous as well as all their derivatives up to a determined order in a certain field of real variables are analytic functions; it is understood, we suppose, there are in the equations only analytic elements. We have for linear equations precise theorems, all the integrals being analytic, if the characteristics are imaginary, and very general propositions have also been obtained in other cases.

The conditions at the limits that one is led to assume are very different according as it is question of an equation of which the integrals are or are not analytic. A type of the first case is given by the problem generalized by Dirichlet; conditions of continuity there play an essential part, and, in general, the solution can not be prolonged from the two sides of the continuum which serves as support to the data; it is no longer the same in the second case, where the disposition of this support in relation to the characteristics plays the principal rôle, and where the field of existence of the solution presents itself under wholly different conditions.

All these notions, difficult to make precise in ordinary language and fundamental for mathematical physics, are not of less interest for infinitesimal geometry.

It will suffice to recall that all the surfaces of constant positive curvature are analytic, while there exist surfaces of constant negative curvature not analytic.

From antiquity has been felt the confused sentiment of a certain economy in natural phenomena; one of the first precise examples is furnished by Fermat's principle relative to the economy of time in the transmission of light.

Then we came to recognize that the general equations of mechanics correspond to a problem of minimum, or more exactly of variation, and thus we obtained the principle of virtual velocities, then Hamilton's principle, and that of least action. A great number of problems appeared then as corresponding to minima of certain definite integrals.

This was a very important advance, because the existence of a minimum could in many cases be regarded as evident, and consequently the demonstration of the existence of the solution was effected.

This reasoning has rendered immense services; the greatest geometers, Gauss in the problem of the distribution of an attracting mass corresponding to a given potential, Riemann in his theory of Abelian functions, have been satisfied with it. To-day our attention has been called to the dangers of this sort of demonstration; it is possible for the minima to be simply limits and not to be actually attained by veritable functions possessing the necessary properties of continuity. We are, therefore, no longer content with the probabilities offered by the reasoning long classic.

Whether we proceed indirectly or whether we seek to give a rigorous proof of the existence of a function corresponding to the minimum, the route is long and arduous.

Further, not the less will it be always useful to connect a question of mechanics or of mathematical physics with a problem of minimum; in this first of all is a source of fecund analytic transformations, and be-sides in the very calculations of the investigation of variations useful indications may appear, relative to the conditions at the limits; a beautiful example of it was given by Kirchoff in the delicate investigation of the conditions at the limits of the equilibrium of flexure of plates.

VI.

I have been led to expand particularly on partial differential equations.

Examples chosen in rational mechanics and in celestial mechanics would readily show the rôle which ordinary differential equations play in the progress of these sciences whose history, as we have seen, has been so narrowly bound to that of analysis.

When the hope of integrating with simple functions was lost, one strove to find developments permitting to follow a phenomenon as long as possible, or at least to obtain information of its qualitative bearing.

For practise, the methods of approximation form an extremely important part of mathematics, and it is thus that the highest parts of theoretic arithmetic find themselves connected with the applied sciences. As to series, the demonstrations themselves of the existence of integrals furnish them from the very first; thus Cauchy's first method gives developments convergent as long as the integrals and the differential coefficients remain continuous.

When any circumstance permits our foreseeing that such is always the case, we obtain developments always convergent. In the problem of n bodies, we can in this way obtain developments valid so long as there are no shocks.

If the bodies, instead of attracting, repel each other, this contingency need not be feared and we would obtain developments valid indefinitely; unhappily, as Fresnel said one day to Laplace, nature is not concerned about analytic difficulties and the celestial bodies attract instead of repelling each other.

One would even be tempted at times to go further than the great physicist and say that nature has sown difficulties in the paths of the analysts.

Thus to take another example, we can generally decide, given a system of differential equations of the first order, whether the general solution is stable or not about a point, and to find developments in series valid for stable solutions it is only necessary that certain inequalities be verified.

But if we apply these results to the equations of dynamics to discuss stability, we find ourselves exactly in the particular case which is unfavorable. Nay, in general here it is not possible to decide on the stability; in the case of a function of forces having a maximum, reasoning classic, but indirect, establishes the stability which can not be deduced from any development valid for every value of the time.

Do not lament these difficulties; they will be the source of future progress.

Such are also the difficulties which still present to us, in spite of so many works, the equations of celestial mechanics; the astronomers have almost drawn from them, since Newton, by means of series practically convergent and approximations happily conducted all that is necessary for the foretelling of the motions of the heavenly bodies.

The analysts would ask more, but they no longer hope to attain the integration by means of simple functions or developments always convergent.

What admirable recent researches have best taught them is the immense difficulty of the problem; a new way has, however, been opened by the study of particular solutions, such as the periodic solutions and the asymptotic solutions which have already been utilized. It is not perhaps so much because of the needs of practice as in order not to avow itself vanquished, that analysis will never resign itself to abandon, without a decisive victory, a subject where it has met so many brilliant triumphs; and again, what more beautiful field could the theories new-born or rejuvenated of the modern doctrine of functions find, to essay their forces, than this classic problem of n bodies?

It is a joy for the analyst to encounter in applications equations that he can integrate with known functions, with transcendents already classed.

Such encounters are unhappily rare; the problem of the pendulum, the classic cases of the motion of a solid body around a fixed point are examples where the elliptic functions have permitted us to effect the integration.

It would also be extremely interesting to encounter a question of mechanics which might be the origin of the discovery of a new transcendent possessing some remarkable property; I would be embarrassed to give an example of it unless in carrying back to the pendulum the début of the theory of elliptic functions.

The interpenetration between theory and applications is here much less than in the questions of mathematical physics. Thus is explained, that, since forty years, the works on ordinary differential equations attached to analytic functions have had in great part a theoretic character altogether abstract.

The pure theory has notably taken the advance; we have had occasion to say that it was well it should be so, but evidently there is here a question of measure, and we may hope to see the old problems profit by the progress accomplished.

It would not be over difficult to give some examples, and I will recall only those linear differential equations, where figure arbitrary parameters whose singular values are roots of entire transcendent functions; which in particular makes the successive harmonics of a vibrating membrane correspond to the poles of a meromorphic function.

It happens also that the theory may be an element of classification in leading to seek conditions for which the solution falls under a determined type, as for example that the integral may be uniform. There have been and there yet will be many interesting discoveries in this way, the case of the motion of a solid heavy body treated by Mme. de Kovalevski and where the Abelian functions were utilized is a remarkable example.

VII.

In studying the reciprocal relations of analysis with mechanics and mathematical physics, we have on our way more than once encountered the infinitesimal geometry, which has proposed so many celebrated problems; in many difficult questions, the happy combination of calculus and synthetic reasonings has realized considerable progress, as show the theories of applicable surfaces and systems triply orthogonal.

It is another part of geometry which plays a grand rôle in certain analytic researches, I mean the geometry of situation or *analysis situs*. We know that Riemann made from this point of view a complete study of the continuum of two dimensions, on which rests his theory of algebraic functions and their integrals.

When this number of dimensions augments, the questions of *analysis situs* become necessarily complicated; the geometric intuition ceases, and the study becomes purely analytic, the mind being guided solely by analogies which may be misleading and need to be discussed very closely. The theory of algebraic functions of two variables, which transports us into a space of four dimensions, without getting from *analysis situs* an aid so fruitful as does the theory of functions of one variable, owes it, however, useful orientations.

There is also another order of questions where the geometry of situation intervenes; in the study of curves traced on a surface and defined by differential equations, the connection of this surface plays an important rôle; this happens for geodesic lines.

The notion of connexity, moreover, presented itself long ago in analysis, when the study of electric currents and magnetism led to non-uniform potentials; in a more general manner certain multiform integrals of some partial differential equations are met in difficult theories, such as that of diffraction, and varied researches must continue in this direction.

From a different point of view, I must yet recall the relations of algebraic analysis with geometry, which manifest themselves so elegantly in the theory of groups of finite order.

A regular polyhedron, say an icosahedron, is on the one hand the solid that all the world knows; it is also, for the analyst, a group of finite order, corresponding to the divers ways of making the polyhedron coincide with itself.

The investigation of all the types of groups of motion of finite order interests not alone the geometers, but also the crystallographers; it goes back essentially to the study of groups of ternary linear substitutions of determinant $+1$, and leads to the thirty-two systems of symmetry of the crystallographers for the particular complex.

The grouping in systems of polyhedra corresponding so as to fill space exhausts all the possibilities in the investigation of the structure of crystals.

Since the epoch when the notion of group was introduced into algebra by Galois, it has taken, in divers ways, considerable development, so that to-day it is met in all parts of mathematics. In the applications, it appears to us above all as an admirable instrument of classification. Whether it is a question of substitution groups or of Sophus Lie's transformation groups, whether it is a question of algebraic equations or of differential equations this comprehensive doctrine permits explanation of the degree of difficulty of the problems treated and teaches how to utilize the special circumstances which present themselves; thus it should interest as well mechanics and mathematical physics as pure analysis.

The degree of development of mechanics and physics has permitted giving to almost all their theories a mathematical form; certain hypotheses, the knowledge of elementary laws, have led to differential relations which constitute the last form under which these theories settle down, at least for a time. These latter have seen little by little their field enlarge with the principles of thermodynamics; to-day chemistry tends to take in its turn a mathematical form.

I will take as witness of it only the celebrated memoir of Gibbs on the equilibrium of chemical systems, so analytic in character, and where it needed some effort on the part of the chemists to recognize, under their algebraic mantle, laws of high importance.

It seems that chemistry has to-day gotten out of the premathematic period, by which every science begins, and that a day must come when will be systematized grand theories, analogous to those of our present mathematical physics, but far more vast, and comprising the ensemble of physicochemic phenomena.

It would be premature to ask if analysis will find in their developments the source of new progress; we do not even know beforehand what analytic types one might find.

I have constantly spoken of differential equations ruling phenomena; will this always be the final form which condenses a theory? Of this I know nothing certain, but we should, however, remember that many hypotheses have been made of nature more or less experimental; among them, one is what has been called the principle of *non-heredity,* which postulates that the future of a system depends only on its present state and its state at an instant infinitely near, or more briefly that accelerations depend only on positions and velocities.

We know that in certain cases this hypothesis is not admissible, at least with the magnitudes directly envisaged; one has sometimes misemployed on this subject the memory of matter, which recalls its past, and has spoken in affected terms of the life of a morsel of steel. Different attempts have been made to give a theory of these phenomena, where a distant past seems to interfere; of them I need not speak here. An analyst may think that in cases so complex it is necessary to abandon the form of differential equations, and resign oneself to envisage *functional equations,* where figure definite integrals which will be the witness of a sort of heredity.

To see the interest which is attached at this moment to functional equations, one might believe in a presentiment of the future needs of science.

VIII.

After having spoken of non-heredity, I scarcely dare touch the question of the applications of analysis to biology.

It will be some time, no doubt, before one forms the functional equations of biologic phenomena; the attempts so far made are in a very modest order of ideas; yet efforts are being made to get out of the purely qualitative field, to introduce quantitative measures. In the question of the variation of certain characteristics, mensuration has been engaged in, and statistic measures which are translated by curves of frequency. The modifications of these curves with successive generations, their decompositions into distinct curves, may give the measure of the stability of species or of the rapidity of mutations, and we know what interest attaches itself to these questions in recent botanic researches. In all this so great is the number of parameters that one questions whether the infinitesimal method itself could be of any service. Some laws of a simple arithmetic character like those of Mendel come occasionally to give renewed confidence in the old aphorism which I cited in the beginning, that all things are explained by numbers; but, in spite of legitimate hopes, it is clear that, in its totality, biology is still far from entering upon a period truly mathematical.

It is not so, according to certain economists, with potential economy. After Cournot, the Lausanne school made an effort extremely interesting to introduce mathematical analysis into political economy.

Under certain hypotheses, which fit at least limiting cases, we find in learned treatises an equation between the quantities of merchandise and their prices, which recalls the equation of virtual velocities in mechanics: this is the equation of economic equilibrium. A function of quantities plays in this theory an essential rôle recalling that of the potential function. Moreover, the best authorized representatives of the school insist on the analogy of economic phenomena with mechanical phenomena. "As rational mechanics, says one of them, considers material points, pure economy considers the homo œconomicus."

Naturally, we find there also the analogues of Lagrange's equations, indispensable matrix of all mechanics.

While admiring these bold works, we fear lest the authors have neglected certain hidden masses, as Helmholtz and Hertz would have said. But although that may happen, there is in these doctrines a curious application of mathematics, which, at least in well-circumscribed cases, has already rendered great services.

I have terminated, messieurs, this summary history of some of the applications of analysis, with the reflections which it has at moments suggested to me.

It is far from being complete; thus I have omitted to speak of the calculus of probabilities, which demands so much subtlety of mind, and of which Pascal refused to explain the niceties to the chevalier de Méré because he was not a geometer.

Its practical utility is of the first rank, its theoretic interest has always been great; it is further augmented to-day, thanks to the importance taken by the researches that Maxwell called *statistical* and which tend to envisage mechanics under a wholly new light.

I hope, however, to have shown, in this sketch, the origin and the reason of the bonds so profound which unite analysis to geometry and physics, more generally to every science bearing on quantities numerically measurable.

The reciprocal influence of analysis and physical theories has been in this regard particularly instructive.

What does the future reserve?

Problems more difficult, corresponding to an approximation of higher order, will introduce complications which we can only vaguely forecast, in speaking, as I have just done, of functional equations replacing systematically our actual differential equations, or further of integrations of equations infinite in number with an infinity of unknown functions. But even though that happens, mathematical analysis will always remain that language which, according to the *mot* of Fourier, has no symbols to express confused notions, a language endowed with an admirable power of transformation and capable of condensing in its formulas a number immense of results.

EMILE PICARD.

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*PRESENT PROBLEMS OF METEOROLOGY.**

NEVER in the history of the science have so many problems presented themselves for solution as at the present time. Numerous *a priori* theories require demonstration and, in fact, the whole structure of meteorology, which has been erected on hypotheses, needs to be strengthened or rebuilt by experimental evidence. Until recently the observations have been carried on at the very bottom of the atmosphere and our predecessors have been compared justly to shell-fish groping about the abysses of the ocean floor to which they are confined.

Probably meteorology had its origin in a crude system of weather predictions, based on signs in the heavens, and it did not become a science until the invention of the principal meteorological instruments in the seventeenth century made possible the study of climatology by the collection of exact and comparable observations at many places on the globe. These data, owing to extensive operations of the meteorological services in the different countries, are now tolerably complete, there being comparatively small portions of the land-surface, at least, for which the climatic elements are not fairly well known, the gaps that remain to be filled lying chiefly on the Antarctic continent and in the interior of Africa.

Although it is about fifty years ago since the first observations, made synchronously

* Address at St. Louis to the Section of Cosmical Physics of the International Congress of Arts and Science.