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## THE SCIENCES OF THE IDEAL.*

I SHALL not attempt, in this address, either to justify or to criticize the name, normative science, under which the doctrines which constitute this division are grouped. It is enough for my purpose to recognize at the outset that I am required, by the plans of this congress, to explain what scientific interests seem to me to be common to the work of the philosophers and of the mathematicians. The task is one which makes severe demands upon the indulgence of the listener, and upon the expository powers of the speaker, but it is a task for which the present age has well prepared the way. The spirit which Descartes and Leibniz illustrated seems likely soon to become, in a new and higher sense, prominent in science. The mathematicians are becoming more and more philosophical. The philosophers, in the near future, will become, I believe, more and more mathematical. It is my office to indicate, as well as the brief time and my poor powers may permit, why this ought to be so.

To this end I shall first point out what is that most general community of interest which unites all the sciences that belong to our division. Then I shall indicate what type of recent and special scientific work most obviously bears upon the tasks of all of us alike. Thirdly, I shall state some results and problems to which this type of scientific work has given rise, and shall try to show what promise we have of an early increase of insight regarding our common interests.

I.

The most general community of interest which unites the various scientific activities

* Address for the St. Louis Congress of Arts and Science, before the Division of Normative Science.

that belong to our division is this: We are all concerned with what may be called ideal truth, as distinct from physical truth. Some of us also have a strong interest in physical truth; but none of us lack a notable and scientific concern for the realm of ideas, viewed as ideas.

Let me explain what I mean by these terms. Whoever studies physical truth (taking that term in its most general sense) seeks to observe, to collate and, in the end, to control, facts which he regards as external to his own thought. But instead of thus looking mainly without, it is possible for a man chiefly to take account, let us say, of the consequences of his own hypothetical assumptions—assumptions which may possess but a very remote relation to the physical world. Or again, it is possible for such a student to be mainly devoted to reflecting upon the formal validity of his own inferences, or upon the meaning of his own presuppositions, or upon the value and the interrelation of human ideals. Any such scientific work, reflective, considerate principally of the thinker's own constructions and purposes, or of the constructions and purposes of humanity in general, is a pursuit of ideal truth. The searcher who is mainly devoted to the inquiry into what he regards as external facts, is indeed active; but his activity is molded by an order of existence which he conceives as complete apart from his activity. He is thoughtful; but a power not himself assigns to him the problems about which he thinks. He is guided by ideals; but his principal ideal takes the form of an acceptance of the world as it is, independently of his ideals. His dealings are with nature. His aim is the conquest of a foreign realm. But the student of what may be called, in general terms, ideal truth, while he is devoted as his fellow, the observer of outer nature, to the general purpose of being faithful to the verity as he finds it, is still aware that his own way of finding, or his own creative activity as an inventor of hypotheses, or his own powers of inference, or his conscious ideals, constitute in the main the object into which he is inquiring, and so form an essential aspect of the sort of verity which he is endeavoring to discover. The guide, then, of such a student is, in a peculiar sense, his own reason. His goal is the comprehension of his own meaning, the conscious and thoughtful conquest of himself. His great enemy is not the mystery of outer nature, but the imperfection of his reflective powers. He is, indeed, as unwilling as is any scientific worker to trust his private caprices. He feels as little as does the observer of outer facts, that he is merely noting down, as they pass, the chance products of his arbitrary fantasy. For him, as for any scientific student, truth is indeed objective; and the standards to which he conforms are eternal. But his method is that of an inner considerateness rather than of a curiosity about external phenomena. His objective world is at the same time an essentially ideal world, and the eternal verity in whose light he seeks to live has, throughout his undertakings, a peculiarly intimate relation to the purposes of his own constructive will.

One may then sum up the difference of attitude which is here in question by saying that, while the student of outer nature is explicitly conforming his plans of action, his ideas, his ideals, to an order of truth which he takes to be foreign to himself—the student of the other sort of truth, here especially in question, is attempting to understand his own plans of action, that is, to develop his ideas, or to define his ideals, or else to do both these things.

Now it is not hard to see that this search for some sort of ideal truth is indeed characteristic of every one of the investigations which have been grouped together in our division of the normative sciences. Pure mathematics shares in common with philosophy this type of scientific interest in ideal, as distinct from physical or phenomenal truth. There is, to be sure, a marked contrast between the ways in which the mathematician and the philosopher approach, select and elaborate their respective sorts of problems. But there is also a close relation between the two types of investigation in question. Let us next consider both the contrast and the analogy in some of their other most general features.

Pure mathematics is concerned with the investigation of the logical consequences of certain exactly statable postulates or hypotheses—such, for instance, as the postulates upon which arithmetic and analysis are founded, or such as the postulates that lie at the basis of any type of geometry. For the pure mathematician, the truth of these hypotheses or postulates depends, not upon the fact that physical nature contains phenomena answering to the postulates, but solely upon the fact that the mathematician is able, with rational consistency, to state these assumed first principles, and to develop their consequences. Dedekind, in his famous essay, 'Was Sind und Was Sollen die Zahlen,' called the whole numbers 'freie Schöpfungen des Menschlichen Geistes'; and, in fact, we need not enter into any discussion of the psychology of our number concept in order to be able to assert that, however we men first came by our conception of the whole numbers, for the mathematician the theory of numerical truth must appear simply as the logical development of the consequences of a few fundamental first principles, such as those which Dedekind himself, or Peano, or other recent writers upon this topic, have, in various forms, stated. A similar formal freedom marks the development of any other theory in the realm of pure mathematics. Pure geometry, from the modern point of view, is neither a doctrine forced upon the human mind by the constitution of any primal form of intuition, nor yet a branch of physical science, limited to describing the spatial arrangement of phenomena in the external world. Pure geometry is the theory of the consequences of certain postulates which the geometer is at liberty consistently to make; so that there are as many types of geometry as there are consistent systems of postulates of that generic type of which the geometer takes account. As is also now well known, it has long been impossible to define pure mathematics as the science of quantity, or to limit the range of the exactly statable hypotheses or postulates with which the mathematician deals to the world of those objects which, ideally speaking, can be viewed as measurable. For the ideally defined measurable objects are by no means the only ones whose properties can be stated in the form of exact postulates or hypotheses; and the possible range of pure mathematics, if taken in the abstract, and viewed apart from any question as to the value of given lines of research, appears to be identical with the whole realm of the consequences of exactly statable ideal hypotheses of every type.

One limitation must, however, be mentioned, to which the assertion just made is, in practise, obviously subject. And this is, indeed, a momentous limitation. The exactly stated ideal hypotheses whose consequences the mathematician develops must possess, as is sometimes said, sufficient intrinsic importance to be worthy of scientific treatment. They must not be trivial hypotheses. The mathematician is not, like the solver of chess problems, merely displaying his skill in dealing with the arbitrary fictions of an ideal game. His truth is, indeed, ideal; his world is, indeed, treated by his science as if this world were the creation of his postulates a 'freie Schöpfung.' But he does not thus create for mere sport. On the contrary, he reports a significant order of truth. As a fact, the ideal systems of the pure mathematician are customarily defined with an obvious, even though often highly abstract and remote, relation to the structure of our ordinary empirical world. Thus the various algebras which have been actually developed have, in the main, definite relations to the structure of the space world of our physical experience. The different systems of ideal geometry, even in all their ideality, still cluster, so to speak, about the suggestions which our daily experience of space and of matter give us. Yet I suppose that no mathematician would be disposed, at the present time, to accept any brief definition of the degree of closeness or remoteness of relation to ordinary experience which shall serve to distinguish a trivial from a genuinely significant branch of mathematical theory. In general a mathematician who is devoted to the theory of functions, or to group theory, appears to spend little time in attempting to show why the development of the consequences of his postulates is a significant enterprise. The concrete mathematical interest of his inquiry sustains him in his labors, and wins for him the sympathy of his fellows. To the questions, 'Why consider the ideal structure of just this system of object at all?' 'Why study various sorts of numbers, or the properties of functions, or of groups, or the system of points in projective geometry?'—the pure mathematician in general, cares to reply only, that the topic of his special investigation appears to him to possess sufficient mathematical interest. The freedom of his science thus justifies his enterprise. Yet, as I just pointed out, this freedom is never mere caprice. This ideal interest is not without a general relation to the concerns even of common sense. In brief, as it seems at once fair to say, the pure mathematician is working under the influence of more or less clearly conscious philosophical motives. He does not usually attempt to define what distinguishes a significant from a trivial system of postulates, or what constitutes a problem worth attacking from the point of view of pure mathematics. But he practically recognizes such a distinction between the trivial and the significant regions of the world of ideal truth, and since philosophy is concerned with the significance of ideas, this recognition brings the mathematician near in spirit to the philosopher.

Such, then, is the position of the pure mathematician. What, by way of contrast, is that of the philosopher? We may reply that to state the formal consequences of exact assumptions is one thing; to reflect upon the mutual relations, and the whole significance of such assumptions, does indeed involve other interests; and these other interests are the ones which directly carry us over to the realm of philosophy. If the theory of numbers belongs to pure mathematics, the study of the place of the number concept in the system of human ideas belongs to philosophy. Like the mathematician, the philosopher deals directly with a realm of ideal truth. But to unify our knowledge, to comprehend its sources, its meaning, and its relations to the whole of human life, these aims constitute the proper goal of the philosopher. In order, however, to accomplish his aims, the philosopher must, indeed, take account of the results of the special physical science; but he must also turn from the world of outer phenomena to an ideal world. For the unity of things is never, for us mortals, anything that we find given in our experience. You can not see the unity of knowledge; you can not describe it as a phenomenon. It is for us now, an ideal. And precisely so, the meaning of things, the relation of knowledge to life, the significance of our ideals, their bearing upon one another—these are never, for us men, phenomenally present data. Hence the philosopher, however much he ought, as indeed he ought, to take account of phenomena, and of the results of the special physical sciences, is quite as deeply interested in his own way, as the mathematician is interested in his way, in the consideration of an ideal realm. Only, unlike the mathematician, the philosopher does not first abstract from the empirical suggestions upon which his exact ideas are actually based, and then content himself merely with developing the logical consequences of these ideas. On the contrary, his main interest is not in any idea or fact in so far as it is viewed by itself, but rather in the interrelations, in the common significance, in the unity, of all fundamental ideas, and in their relations both to the phenomenal facts and to life! On the whole, he, therefore, neither consents, like the student of a special science of experience, to seek his freedom solely through conformity to the phenomena which are to be described; nor is he content, like the pure mathematician, to win his truth solely through the exact definition of the formal consequences of his freely defined hypotheses. He is making an effort to discover the sense and the unity of the business of his own life.

It is no part of my purpose to attempt to show here how this general philosophical interest differentiates into the various interests of metaphysics, of the philosophy of religion, of ethics, of esthetics, of logic. Enough—I have tried to illustrate how, while both the philosopher and the mathematician have an interest in the meaning of ideas rather than in the description of external facts, still there is a contrast which does, indeed, keep their work in large measure asunder, viz., the contrast due to the fact that the mathematician is directly concerned with developing the consequences of certain freely assumed systems of postulates or hypotheses; while the philosopher is interested in the significance, in the unity and in the relation to life, of all the fundamental ideals and postulates of the human mind.

Yet not even thus do we sufficiently state how closely related the two tasks are. For this very contrast, as we have also suggested, is, even within its own limits, no final or perfectly sharp contrast. There is a deep analogy between the two tasks. For the mathematician, as we have just seen, is not evenly interested in developing the consequences of any and every system of freely assumed postulates. He is no mere solver of arbitrary ideal puzzles in general. His systems of postulates are so chosen as to be not trivial, but significant. They are, therefore, in fact, but abstractly defined aspects of the very system of eternal truth whose expression is the universe. In this sense the mathematician is as genuinely interested as is the philosopher in the significant use of his scientific freedom. On the other hand, the philosopher, in reflecting upon the significance and the unity of fundamental ideas, can only do so with success in case he makes due inquiry into the logical consequences of given ideas. And this he can accomplish only if, upon occasion, he employs the exact methods of the mathematician, and develops his systems of ideal truth with the precision of which only mathematical research is capable. As a fact, then, the mathematician and the philosopher deal with ideal truth in ways which are not only contrasted, but profoundly interconnected. The mathematician, in so far as he consciously distinguishes significant from trivial problems, and ideal systems, is a philosopher. The philosopher, in so far as he seeks exactness of logical method, in his reflection, must meanwhile aim to be, within his own limits, a mathematician. He, indeed, will not in future, like Spinoza, seek to reduce philosophy to the mere development, in mathematical form, of the consequences of certain arbitrary hypotheses. He will distinguish between a reflection upon the unity of the system of truth and an abstract development of this or that selected aspect of the system. But he will see more and more that, in so far as he undertakes to be exact, he must aim to become, in his own way, and with due regard to his own purposes, mathematical; and thus the union of mathematical and philosophical inquiries, in the future, will tend to become closer and closer.

II.

So far, then, I have dwelt upon extremely general considerations relating to the unity and the contrast of mathematical and philosophical inquiries. I can well conceive, however, that the individual worker in any one of the numerous branches of investigation which are represented by the body of students whom I am privileged to address, may at this point mentally interpose the objection that all these considerations are, indeed, far too general to be of practical interest to any of us. Of course, all we who study these so-called normative sciences are, indeed, interested in ideas, for their own sakes—in ideas so distinct from, although of course also somehow related to, phenomena. Of course some of us are rather devoted to the development of the consequences of exactly stated ideal hypotheses, and others to reflecting as we can upon what certain ideas and ideals are good for, and upon what the unity is of all ideas and ideals. Of course if we are wise enough to do so, we have much to learn from one another. But, you will say, the assertion of all these things is a commonplace. The expression of the desire for further mutual cooperation is a pious wish. You will insist upon asking further: "Is there just now any concrete instance in a modern type of research which furnishes results such as are of interest to all of us? Are we actually doing any productive work in common? Are the philosophers contributing anything to human knowledge which has a genuine bearing upon the interests of mathematical science? Are the mathematicians contributing anything to philosophy?"

These questions are perfectly fair. Moreover, as it happens, they can be distinctly answered in the affirmative. The present age is one of a rapid advance in the actual unification of the fields of investigation which are included within the scope of this present division. What little time remains to me must be devoted to indicating, as well as I can, in what sense this is true. I shall have still to deal in very broad generalities. I shall try to make these generalities definite enough to be not wholly unfruitful.

We have already emphasized one question which may be said to interest, in a very direct way, both the mathematician and the philosopher. The ideal postulates, whose consequences mathematical science undertakes to develop, must be, we have said, significant postulates, involving ideas whose exact definition and exposition repay the labor of scientific scrutiny. Number, space, continuity, functional correspondence or dependence, group-structure—these are examples of such significant ideas; the postulates or ideal assumptions upon which the theory of such ideas depends are significant postulates, and are not the mere conventions of an arbitrary game. But now what constitutes the significance of an idea, or of an abstract mathematical theory? What gives an idea a worthy place in the whole scheme of human ideas? Is it the possibility of finding a physical application for a mathematical theory which for us decides what is the value of the theory? No, the theory of functions, the theory of numbers, group theory, have a significance which no mathematician would consent to measure in terms of the present applicability or non-applicability of these theories in physical science? In vain, then, does one attempt to use the test of applied mathematics as the main criticism of the value of a theory of pure mathematics. The value of an idea, for the sciences which constitute our division, is dependent upon the place which this idea occupies in the whole organized scheme or system of human ideas. The idea of number, for instance, familiar as its applications are, does not derive its main value from the fact that eggs and dollars and star-clusters can be counted, but rather from the fact that the idea of numbers has those relations to other fundamental ideas which recent logical theory has made prominent—relations, for instance, to the concept of order, to the theory of classes or collections of objects viewed in general, and to the metaphysical concept of the self. Relations of this sort, which the discussions of the number concept by Dedekind, Cantor, Peano and Russell have recently brought to light—such relations, I say, constitute what truly justified Gauss in calling the theory of numbers a 'divine science.' As against such deeper relations, the countless applications of the number concept in ordinary life, and in science, are, from the truly philosophical point of view, of comparatively small moment. What we want, in the work of our division of the sciences, is to bring to light the unity of truth, either, as in mathematics, by developing systems of truth which are significant by virtue of their actual relations to this unity, or, as in philosophy, by explicitly seeking the central idea about which all the many ideas cluster.

Now, an ancient and fundamental problem for the philosophers is that which has been called the problem of the categories. This problem of the categories is simply the more formal aspect of the whole philosophical problem just defined. The philosopher aims to comprehend the unity of the system of human ideas and ideals. Well, then, what are the primal ideas? Upon what group of concepts do the other concepts of human science logically depend? About what central interests is the system of human ideals clustered? In ancient thought Aristotle already approached this problem in one way. Kant, in the eighteenth century, dealt with it in another. We students of philosophy are accustomed to regret what we call the excessive formalism of Kant, to lament that Kant was so much the slave of his own relatively superficial and accidental table of categories, and that he made the treatment of every sort of philosophical problem turn upon his own schematism. Yet we can not doubt that Kant was right in maintaining that philosophy needs, for the successful development of every one of its departments, a well-devised and substantially complete system of categories. Our objection to Kant's over-confidence in the virtues of his own schematism is due to the fact that we do not now accept his table of categories as an adequate view of the fundamental concepts. The efforts of philosophers since Kant have been repeatedly devoted to the task of replacing his scheme of categories by a more adequate one. I am far from regarding these purely philosophical efforts made since Kant as fruitless, but they have remained, so far, very incomplete, and they have been held back from their due fulness of success by the lack of a sufficiently careful survey and analysis of the processes of thought as these have come to be embodied in the living sciences. Such concepts as number, quantity, space, time, cause, continuity, have been dealt with by the pure philosophers far too summarily and superficially. A more thoroughgoing analysis has been needed.

But now, in comparatively recent times, there has developed a region of inquiry which one may call by the general name of modern logic. To the constitution of this new region of inquiry men have principally contributed who began as mathematicians, but who, in the course of their work, have been led to become more and more philosophers. Of late, however, various philosophers, who were originally in no sense mathematicians, becoming aware of the importance of the new type of research, are in their turn attempting both to assimilate and to supplement the undertakings which were begun from the mathematical side. As a result, the logical problem of the categories has to-day become almost equally a problem for the logicians of mathematics and for those students of philosophy who take any serious interest in exactness of method in their own branch of work. The result of this actual cooperation of men from both sides is that, as I think, we are to-day, for the first time, in sight of what is still, as I freely admit, a somewhat distant goal, viz., the relatively complete rational analysis and tabulation of the fundamental categories of human thought. That the student of ethics is as much interested in such an investigation as is the metaphysician, that the philosopher of religion needs a well-completed table of categories quite as much as does the pure logician, every competent student of such topics ought to admit. And that the enterprise in question keenly interests the mathematicians is shown by the prominent part which some of them have taken in the researches in question. Here, then, is the type of recent scientific work whose results most obviously bear upon the tasks of all of us alike.

A catalogue of the names of the workers in this wide field of modern logic would be out of place here. Yet one must, indeed, indicate what lines of research are especially in question. From the purely mathematical side, the investigations of the type to which I now refer may be viewed (somewhat arbitrarily) as beginning with that famous examination into one of the postulates of Euclid's geometry which gave rise to the so-called non-Euclidean geometry. The question here originally at issue was one of a comparatively limited scope, viz., the question whether Euclid's parallel-line postulate was a logical consequence of the other geometrical principles. But the investigation rapidly develops into a general study of the foundations of geometry—a study to which contributions are still almost constantly appearing. Somewhat independently of this line of inquiry there grew up, during the latter half of the nineteenth century, that reexamination of the bases of arithmetic and analysis which is associated with the names of Dedekind, Weierstrass and George Cantor. At the present time, the labors of a number of other inquirers (amongst whom we may mention the school of Peano and Pieri in Italy, and men such as Poincaré and Couturat in France, Hilbert in Germany, Bertram Russell and Whitehead in England and an energetic group of our American mathematicians—men such as Professor Moore, Professor Halsted, Dr. Huntington, Dr. Veblen and a considerable number of others) have been added to the earlier researches. The result is that we have recently come for the first time to be able to see, with some completeness, what the assumed first principles of pure mathematics actually are. As was to be expected, these principles are capable of more than one formulation, according as they are approached from one side or from another. As was also to be expected, the entire edifice of pure mathematics, so far as it has yet been erected, actually rests upon a very few fundamental concepts and postulates, however you may formulate them. What was not observed, however, by the earlier, and especially by the philosophical, students of the categories, is the form which these postulates tend to assume when they are rigidly analyzed.

This form depends upon the precise definition and classification of certain types of relations. The whole of geometry, for instance, including metrical geometry, can be developed from a set of postulates which demand the existence of points that stand in certain ordinal relationships. The ordinal relationships can be reduced, according as the series of points considered is open or closed, either to the well-known relationship in which three points stand when one is between the other two upon a right line, or else to the ordinal relationship in which four points stand when they are separated by pairs; and these two ordinal relationships, by means of various logical devices, can be regarded as variations of a single fundamental form. Cayley and Klein founded the logical theory of geometry here in question. Russell, and in another way Dr. Veblen, have given it its most recent expressions. In the same way, the theory of whole numbers can be reduced to sets of principles which demand the existence of certain ideal objects in certain simple ordinal relations. Dedekind and Peano have worked out such ordinal theories of the number concept. In another development of the theory of the cardinal whole numbers, which Russell and Whitehead have worked out, ordinal concepts are introduced only secondarily, and the theory depends upon the fundamental relation of the equivalence or non-equivalence of collections of objects. But here also a certain simple type of relation determines the definitions and the development of the whole theory.

Two results follow from such a fashion of logically analyzing the first principles of mathematical science. In the first place, as just pointed out, we learn *how few and simple are the conceptions and postulates* upon which the actual edifice of exact science rests. Pure mathematics, we have said, is free to assume what it chooses. Yet the assumptions whose presence as the foundation principles of the actually existent pure mathematics an exhaustive examination thus reveals, show by their fewness that the ideal freedom of the mathematician to assume and to construct what he pleases, is indeed, in practise, a very decidedly limited freedom. The limitation is, as we have already seen, a limitation which has to do with the essential significance of the fundamental concepts in question. And so the result of this analysis of the bases of the actually developed and significant branches of mathematics, constitutes a sort of empirical revelation of what categories the exact sciences have practically found to be of such significance as to be worthy of exhaustive treatment. Thus the instinctive sense for significant truth which has all along been guiding the development of mathematics, comes at least to a clear and philosophical consciousness. And meanwhile the essential categories of thought are seen in a new light.

The second result still more directly concerns a philosophical logic. It is this: Since the few types of relations which this sort of analysis reveals as the fundamental ones in exact science are of such importance, the logic of the present day is especially required to face the questions: *What is the nature of our concept of relations?* What are the various possible types of relations? Upon what does the variety of these types depend? What unity lies beneath the variety?

As a fact, logic, in its modern forms, viz., first that symbolic logic which Boole first formulated, which Mr. Charles S. Peirce and his pupils have in this country already so highly developed, and which Schroeder in Germany, Peano's school in Italy and a number of recent English writers, have so effectively furthered—and secondly, the logic of scientific method, which is now so actively pursued, in France, in Germany and in the English-speaking countries—this whole movement in modern logic, as I hold, is rapidly approaching *new solutions of the problem of the fundamental nature and the logic of relations.* The problem is one in which we are all equally interested. To De Morgan in England, in an earlier generation, and, in our time, to Charles Peirce in this country, very important stages in the growth of these problems are due. Russell, in his work on the 'Principles of Mathematics,' has very lately undertaken to sum up the results of the logic of relations, as thus far developed, and to add his own interpretations. Yet I think that Russell has failed to get as near to the foundations of the theory of relations as the present state of the discussion permits. For Russell has failed to take account of what I hold to be the most fundamentally important generalization yet reached in the general theory of relations. This is the generalization set forth as early as 1890, by Mr. A. B. Kempe, of London, in a pair of wonderful, but too much neglected, papers, entitled, respectively, 'The Theory of Mathematical Form,' and 'The Analogy between the Logical Theory of Classes and the Geometrical Theory of Points.' A mere hint first as to the more precise formulation of the problem at issue, and then later as to Kempe's special contribution to that problem, may be in order here, despite the impossibility of any adequate statement.

III.

The two most obviously and universally important kinds of relations known to the exact sciences, as these sciences at present exist, are: (1) The relations of the type of equality or equivalence; and (2) the relations of the type of before and after, or greater and less. The first of these two classes of relations, viz., the class represented, although by no means exhausted, by the various relations actually called, in different branches of science by the one name equality, this class I say, might well be named, as I myself have proposed, the leveling relations. A collection of objects between any two of which some one relation of this type holds, may be said to be a collection whose members, in some defined sense or other, are on the same level. The second of these two classes of relations, viz., those of the type of before and after, or greater and less—this class of relations, I say, consists of what are nowadays often called the serial relations. And a collection of objects such that, if any pair of these objects be chosen, a determinate one of this pair stands to the other one of the same pair in some determinate relation of this second type, and in a relation which remains constant for all the pairs that can be thus formed out of the members of this collection—any such collection, I say, constitutes a one-dimensional open series. Thus, in case of a file of men, if you choose any pair of men belonging to the file, a determinate one of them is, in the file, before the other. In the number series, of any two numbers, a determinate one is greater than the other. Wherever such a state of affairs exists, one has a series.

Now these two classes of relations, the leveling relations and the serial relations, agree with one another, and differ from one another in very momentous ways. They *agree* with one another in that both the leveling and the serial relations are what is technically called *transitive;* that is, both classes conform to what Professor James has called the law of 'skipped intermediaries.' Thus, if $A$ is equal to $B$, and $B$ is equal to $C$, it follows that $A$ is equal to $C$. If $A$ is before $B$, and $B$ is before $C$, then $A$ is before $C$. And this property, which enables you in your reasonings about these relations to skip middle terms, and so to perform some operation of elimination, is the property which is meant when one calls relations of this type transitive. But, on the other hand, these two classes of relations *differ* from each other in that the leveling relations are, while the serial relations are not, *symmetrical* or reciprocal. Thus, if $A$ is equal to $B$, $B$ is equal to $A$. But if $X$ is greater than $Y$, then $Y$ is not greater than $X$, but less than $X$. So the leveling relations are symmetrical transitive relations. But the serial relations are transitive relations which are not symmetrical.

All this is now well known. It is notable, however, that nearly all the processes of our exact sciences, as at present developed, can be said to be essentially such as lead either to the placing of sets or classes of objects on the same level, by means of the use of symmetrical transitive relations, or else to the arranging of objects in orderly rows or series, by means of the use of transitive relations which are not symmetrical. This holds also of all the applications of the exact sciences. Whatever else you do in science (or, for that matter, in art), you always lead, in the end, either to the arranging of objects, or of ideas, or of acts, or of movements, in rows or series, or else to the placing of objects or ideas of some sort on the same level, by virtue of some equivalence, or of some invariant character. Thus numbers, functions, lines in geometry, give you examples of serial relations. Equations in mathematics are classic instances of leveling relations. So, of course, are invariants. Thus, again, the whole modern theory of energy consists of two parts, one of which has to do with levels of energy, in so far as the quantity of energy of a closed system remains invariant through all the transformations of the system, while the other part has to do with the irreversible serial order of the transformations of energy themselves, which follow a set of unsymmetrical relations, in so far as energy tends to fall from higher to lower levels of intensity within the same system.

The entire conceivable universe then, and all of our present exact science, can be viewed, if you choose, as a collection of objects or of ideas that, whatever other types of relations may exist, are at least largely characterized either by the leveling relations, or by the serial relations, or by complexes of both sorts of relations. Here, then, we are plainly dealing with very fundamental categories. The 'between' relations of geometry can of course be defined, if you choose, in terms of transitive relations that are not symmetrical. There are, to be sure, some other relations present in exact science, but the two types, the serial and leveling relations, are especially notable.

So far the modern logicians have for some time been in substantial agreement. Russell's brilliant book is a development of the logic of mathematics very largely in terms of the two types of relations which, in my own way, I have just characterized; although Russell gives due regard, of course, to certain other types of relations.

But hereupon the question arises, 'Are these two types of relations what Russell holds them to be, viz., ultimate and irreducible logical facts, unanalyzable categories—mere data for the thinker? Or can we reduce them still further, and thus simplify yet again our view of the categories?

Here is where Kempe's generalization begins to come into sight. These two categories, in at least one very fundamental realm of exact thought, can be reduced to one. There is, namely, a world of ideal objects which especially interest the logician. It is the world of a *totality of pos-*

*sible logical classes,* or again, it is the ideal world, equivalent in formal structure to the foregoing, but composed of a *totality of possible statements,* or thirdly, it is the world, equivalent once more, in formal structure, to the foregoing, but consisting of a *totality of possible acts of will,* of possible decisions. When we proceed to consider the relational structure of such a world, taken merely in the abstract as such a structure, a relation comes into sight which at once appears to be peculiarly general in its nature. It is the so-called illative relation, the relation which obtains between two classes when one is subsumed under the other, or between two statements, or two decisions, when one implies or entails the other. This relation is transitive, but may be either symmetrical or not symmetrical; so that, according as it is symmetrical or not, it may be used either to establish levels or to generate series. In the order system of the logician's world, the relational structure is thus, in any case, a highly general and fundamental one.

But this is not all. In this the logician's world of classes, or of statements, or of decisions, there is also another relation observable. This is the relation of exclusion or mutual opposition. This is a purely symmetrical or reciprocal relation. It has two forms—obverse or contradictory opposition, *i. e.,* negation proper, and contrary opposition. But both these forms are purely symmetrical. And by proper devices each of them can be stated in terms of the other, or reduced to the other. And further, as Kempe incidentally shows, and as Mrs. Ladd Franklin has also substantially shown in her important theory of the syllogism, *it is possible to state every proposition, or complex of propositions involving the illative relation, in terms of this purely symmetrical relation of opposition.* Hence, so far as mere relational form is concerned, the illative relation itself may be wholly reduced to the symmetrical relation of opposition. This is our first result as to the relational structure of the realm of pure logic, *i. e.,* the realm of classes, of statements, or of decisions.

It follows that, in describing the logician's world of possible classes or of possible decisions, *all unsymmetrical, and so all serial, relations can be stated solely in terms of symmetrical relations, and can be entirely reduced to such relations.* Moreover, as Kempe has also very prettily shown, the relation of opposition, in its two forms, just mentioned, need not be interpreted as obtaining merely between pairs of objects. It may and does obtain between triads, tetrads, $n$-ads of logical entities; and so all that is true of the relations of logical classes may consequently be stated merely by ascribing certain perfectly symmetrical and homogeneous predicates to pairs, triads, tetrads, $n$-ads of logical objects. The essential contrast between symmetrical and unsymmetrical relations thus, in this ideal realm of the logician, simply vanishes. The categories of the logician's world of classes, of statements, or of decisions, are marvelously simple. All the relations present may be viewed as variations of the mere conception of opposition as distinct from non-opposition.

All this holds, of course, so far, merely for the logician's world of classes or of decisions. There, at least, all serial order can actually be derived from wholly symmetrical relations. But Kempe now very beautifully shows (and here lies his great and original contribution to our topic)—he shows, I say, that the ordinal relations of geometry, as well as of the number-system, can all be regarded as indistinguishable from *mere variations of those relations which, in pure logic, one finds to be the symmetrical relations obtaining within pairs or triads of classes or of statements.*

The formal identity of the geometrical relation called 'between' with a purely logical relation which one can define as existing or as not existing amongst the members of a given triad of logical classes, or of logical statements, is shown by Kempe in a fashion that I can not here attempt to expound. But Kempe's result thus enables one, as I believe, to simplify the theory of relations far beyond the point which Russell, in his brilliant book has reached. For Kempe's triadic relation in question can be stated, in what he calls its obverse form, in perfectly symmetrical terms. And he proves very exactly that the resulting logical relation is precisely identical, in all its properties, with the fundamental ordinal relation of geometry.

Thus the order-systems of geometry and analysis appear simply as special cases of the more general order-system of pure logic. The whole, both of analysis and of geometry, can be regarded as a description of certain selected groups of entities, which are chosen, according to special rules, from a single ideal world. This general and inclusive ideal world consists simply of *all the objects which can stand to one another in those symmetrical relations wherein the pure logician finds various statements, or various decisions inevitably standing,* 'Let me,' says in substance Kempe, 'choose from the logician's ideal world of classes or decisions, what entities I will; and I will show you a collection of objects that are in their relational structure, precisely identical with the points of a geometer's space of $n$ dimensions.' In other words, all of the geometer's figures and relations can be precisely pictured by the relational structure of a selected system of classes or of statements, whose relations are wholly and explicitly logical relations, such as opposition, and whose relations may all be regarded, accordingly, as reducible to a single type of purely symmetrical relation.

Thus, for *all* exact science, and not merely for the logician's special realm, the contrast between symmetrical and unsymmetrical relations proves to be, after all, superficial and derived. The purely logical categories, such as opposition, and such as hold within the calculus of statements, are, apparently, the basal categories of all the exact science that has yet been developed. Series and levels are relational structures that, sharply as they are contrasted, can be derived from a single root.

I have restated Kempe's generalization in my own way. I think it the most promising step towards new light as to the categories that we have made for some generations.

In the field of modern logic, I say, then, work is doing which is rapidly tending towards the unification of the tasks of our entire division. For this problem of the categories, in all its abstractness, is still a common problem for all of us. Do you ask, however, what such researches can do to furnish more special aid to the workers in metaphysics, in the philosophy of religion, in ethics, or in esthetics, beyond merely helping towards the formulation of a table of categories—then I reply that we are already not without evidence that such general researches, abstract though they may seem, are bearing fruits which have much more than a merely special interest. Apart from its most general problems, that analysis of mathematical concepts to which I have referred has in any case revealed numerous unexpected connections between departments of thought which had seemed to be very widely sundered. One instance of such a connection I myself have elsewhere discussed at length, in its general metaphysical bearings. I refer to the logical identity which Dedekind first pointed out between the mathematical concept of the ordinal number of series and the philosophical concept of the formal structure of an ideally completed self. I have maintained that this formal identity throws light upon problems which have as genuine an interest for the student of the philosophy of religion as for the logician of arithmetic. In the same connection it may be remarked that, as Couturat and Russell, amongst other writers, have very clearly and beautifully shown, the argument of the Kantian mathematical antinomies needs to be explicitly and totally revised in the light of Cantor's modern theory of infinite collections. To pass at once to another, and a very different instance: The modern mathematical conceptions of what is called group theory have already received very wide and significant applications, and promise to bring into unity regions of research which, until recently, appeared to have little or nothing to do with one another. Quite lately, however, there are signs that group theory will soon prove to be of importance for the definition of some of the fundamental concepts of that most refractory branch of philosophical inquiry, esthetics. Dr. Emch, in an important paper in the *Monist,* called attention, some time since, to the symmetry groups to which certain esthetically pleasing forms belong, and endeavored to point out the empirical relations between these groups and the esthetic effects in question. The grounds for such a connection between the groups in question and the observed esthetic effects, seemed, in the paper of Dr. Emch to be left largely in the dark. But certain papers recently published in the country by Miss Ethel Puffer, bearing upon the psychology of the beautiful (although the author has approached the subject without being in the least consciously influenced, as I understand, by the conceptions of the mathematical group theory), still actually lead, if I correctly grasp the writer's meaning, to the doctrine that the esthetic object, viewed as a psychological whole, must possess a structure closely, if not precisely, equivalent to the ideal structure of what the mathematician calls a group. I myself have no authority regarding esthetic concepts, and speak subject to correction. But the unexpected, and in case of Miss Puffer's research, quite unintended, appearance of group theory in recent esthetic analysis is to me an impressive instance of the use of relatively new mathematical conceptions in philosophical regions which *seem,* at first sight, very remote from mathematics.

That both the group concept and the concept of the self just suggested are sure to have also a wide application in the ethics of the future, I am myself well convinced. In fact, no branch of philosophy is without close relations to all such studies of fundamental categories.

These are but hints and examples. They suffice, I hope, to show that the workers in this division have deep common interests, and will do well, in future, to study the arts of cooperation, and to regard one another's progress with a watchful and cordial sympathy. In a word: Our common problem is the theory of the categories. That problem can be solved only by the cooperation of the mathematicians and of the philosophers.

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## SCIENTIFIC BOOKS.

*The Harriman Alaska Expedition.* Vol. X. Crustaceans. By MARY J. RATHBUN, HARRIET RICHARDSON, S. J. HOLMES and LEON J. COLE. New York, Doubleday, Page and Co. 1904. Pp. x + 337. 8vo; with xxvi plates and 128 figures in the text.

In working out the shrimps of the Harriman expedition Miss Rathbun was obliged to review the entire material of that group from northwest America which had accumulated in the National Museum and, in addition to the