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THE LOBACHEVSKI PRIZE.

THE third award of the great Lobachevski prize was the occasion for considering particularly the achievements of two men during the past five years. The first of these is Professor Hilbert, of Goettingen; the second, Professor Barbarin, of Bordeaux.

The committee asked from the most distinguished of French mathematicians, Poincaré, a report on those works of Hilbert relevant to the decision.

With French thrift, Poincaré used as the greater part of this report a review of Hilbert's 'Grundlagen' which had already been published three times, two parts of which I quoted in my St. Louis address, to point out two errors (SCIENCE, N. S., Vol. XIX., No. 480, pp. 401–413). But the works to be considered included others which had only appeared after that review was written, so that Poincaré was compelled to recast and supplement this review of his, and some of these additions are of high interest.

Our ideas, he says, on the origin and scope of geometric verities have, since a century, evolved in a very rapid way. The creations of Lobachevski, of Bolyai, of Riemann have inaugurated a new era. Certes they have not discouraged the men, only too numerous, who seek to demonstrate the postulatum of Euclid. These, alas! nothing could discourage. But they have convinced all the true savants of the inanity of such an attempt. This was the first result of the invention of the non-Euclidean geometries. Lie pushed the matter farther, but his spaces were all assumed by him to be number-manifolds.

A new progress remained to be accomplished, and the honor of this belongs to Hilbert. It is important, however, to say a word of the works which have prepared and rendered possible this advance.

Since the time of Lobachevski, mathematical thought has undergone a profound evolution, not alone in geometry, but in arithmetic and analysis. The notion of number has been made more clear and precise; at the same time it has received diverse generalizations. The most precious for the analyst is that which results from the introduction of neomonics, which modern mathematicians could not now dispense with.

Again, many Italian geometers, such as Peano and Padoa, have created a *pasigraphie,* that is to say, a sort of universal algebra where all reasonings are replaced by symbols or formulas.

Finally must be cited the book of Veronese on the foundations of geometry, where the author applies for the first time to geometry the transfinite numbers of Georg Cantor.

In 1899 Hilbert published a memoir entitled 'Grundlagen der Geometrie,' full of ideas the most original. Moreover, this was not the first time he had occupied himself with analogous questions, witness his letter of 1894 to Felix Klein: 'Ueber die gerade Linie als kürzeste Verbindung zweier Punkte.' He has since published in divers journals a series of articles entitled: 'On the theorem of the equality of the basal angles in the isosceles triangle'; 'New founding of the Bolyai-Lobachevski geometry'; 'On the foundations of geometry'; 'On surfaces of constant Gaussian curvature.'

All these articles have been united in a second edition of his jubilee memoir; and I must add that this second edition contains a series of improvements and additions which greatly augment its value.

It is, therefore, this second edition that we will follow in our analysis; but we will join with it, on the one hand, other works of Hilbert, such as his article 'Ueber den Zahlbegriff' and his Paris address on the mathematical problems of the future, and, on the other hand, many theses written by his scholars, under his direct inspiration, and which consequently aid us in comprehending his thought. The principal are: 'Ueber die Geometrieen in denen die Geraden die kürzesten sind' by Georg Hamel, and 'Die Legendre'schen Sätze ueber die Winkelsumme im Dreieck' by M. Dehn.

"Hilbert commences," continues Poincaré, "by establishing the complete list of axioms, straining not to forget one.

"Is his list final? It is permitted to believe so, since it seems to have been drawn up with care."

So says Poincaré for the fourth time. But if Hilbert's receiving the Lobachevski prize depended on his list of axioms being '*définitive,*' it could not be given to him. A young pupil of my own, R. L. Moore, by a charmingly simple proof has abolished the ugliest of the list, and Hilbert has already acknowledged the redundancy. Another point Hilbert himself changed in the French translation of his 'Festschrift' by Laugel. Poincaré had said: "The axiom that the sect AB is congruent to the inverse sect BA (which implies the symmetry of space) is not identical with those which are explicitly stated. I do not know whether it could be logically deduced from them; I believe it could."

In his 'Report,' Poincaré now says: "An important point is not here treated; the list of axioms should be completed by saying that the sect AB is congruent to the inverse sect BA.

"This axiom implies the symmetry of

space and the equality of the angles at the base in an isosceles triangle. Hilbert does not here treat this question, but he has made it the object of a memoir to which we will recur." This is a mistake. Hilbert explicitly assumes $AB \equiv BA$, but the equality of the basal angles does not follow therefrom.

It used to be supposed that the Euclidean straight was of essence continuous, and this putative continuity was rested upon to give continuity to the real number system. This mistake is made by Professor H. B. Fine in his book 'The Number-system of Algebra.' For example, on page 43 he says: "*The entire system of real numbers,* however, inasmuch at it contains an individual number to correspond to every individual point in the continuous series of points forming a right line, *is continuous.*"

Dedekind had long ago called attention to the fact that Euclid's space had no need of continuity. In an article of my own 'How the new mathematics interprets the old,' March 4, 1893, is a quotation of his construction of a discrete space, which goes on, "yet despite the discontinuity, the perforation, of this space, all constructions occurring in Euclid are in it just as achievable as in perfectly continuous space. The discontinuity of this space would, therefore, never be noticed, never be discovered, in Euclid's science. Um so schoener erscheint es mir, das der Mensch ohne jede Vorstellung von messbaren Groessen, und zwar durch ein endliches System einfacher Denkschritte sich zur Schoepfung des reinen, stetigen Zahlenreiches aufschwingen kann; und erst mit diesem Hülfsmittel wird es ihm nach meiner Ansicht moeglich, die Vorstellung vom stetigen Raume zu einer deutlich auszubilden."

There are naturally no points on the Euclidean straight to correspond to the series of irrational numbers, and Euclid felt no more ambition to have them there than he did to have a set of automobiles, and for the same reason, irrational numbers and automobiles had not yet been created.

Hilbert's axioms, analyzing Euclid's space, did not make it continuous. Poincaré called attention to this, and spoke of the space burdened with these irrational points as *our* space in contrast to Euclid's space, as if we were debarred by modernity from living in the splendidly free and disjointed space of the glorious old Alexandrian, who spurned the idea of any other way even for kings. "In Hilbert's space," he says, "there are not all the points which are in ours, but only those that one could construct, starting from two given points, by means of the ruler and compasses. In this space, for example, there would not exist, in general, an angle which would be the third part of a given angle.

"I have no doubt that this conception would have been regarded by Euclid as more rational than ours."

He then proceeds, following Dedekind, to give an assumption which will lug in these irrational points. Hilbert in Laugel's translation did the same, but by a quite different assumption, which he calls the 'Axiom der Vollständigkeit.'

"Note. We remark that to the five preceding groups of axioms one may still add the following axiom which is not of a purely geometric nature and which, from the theoretical point of view, merits particular attention.

"AXIOM OF COMPLETENESS.

"*To the system of points, straights and planes, it is impossible to add other beings (êtres) so that the system thus generalized forms a new geometry where the axioms of the five groups I.–V. are all verified. In other words: the elements of geometry form a system of beings which, if one maintains* *all the axioms, is not susceptible of any extension.*''

In speaking of the non-Archimedean geometry, the 'Review' made no mention of Veronese. The 'Report,' however, says: ''In this conception, so audacious, Hilbert had had a precursor. In his foundations of geometry Veronese had had an analogous idea. Chapter VI. of his introduction is the development of a veritable arithmetic and of a veritable geometry non-Archimedean where the transfinite numbers of Cantor play a preponderant rôle. Nevertheless, by the elegance and the simplicity of his exposition, by the depth of his philosophic views, by the advantage he has obtained from the fundamental idea, Hilbert has made the new geometry his own.''

The 'Report' is in 39 pages. The incorporation of the already published 'Review' stops with page 25. The last fourteen pages are entirely new, as follows: The memoir we have just analyzed puts in evidence the importance of the new non-Archimedean geometry. It discusses the rôle of the axiom of Archimedes in geometric reasoning; and the principal result of this discussion may be summed up thus: If we abandon this axiom and retain only the axioms of the first four groups, the essential results of Euclidean geometry are not altered; but this is not so if one retains only the projective axioms [assumptions of association] and those of order [betweenness], together with the postulatum of Euclid, but abandons at the same time the axiom of Archimedes and the metric axioms [assumptions of congruence]; we come then to the non-Pascalian geometry.

Then comes the question, does this that we have just said of the Euclidean geometry remain true of the Lobachevskian?

In other words, if we preserve only the axioms of the first three groups (projective, of order and metric) and replace the postulatum of Euclid by that of Lobachevski, shall we arrive at the fundamental theorems of Lobachevski *without using the axiom of Archimedes?*

This is the question that Hilbert has settled in his article 'Ueber eine neue Begründung der Bolyai-Lobatschefskyschen Geometrie.'

He answers it affirmatively and shows in particular that there always exists a common perpendicular to two straights of the plane which do not meet without being parallel.

I would call attention to the statement of the postulate of Lobachevski: ''If b is any straight of the plane and A a point not situated on this straight, there pass always through A two demi-straights [rays], a_1 and a_2, which are not in the prolongation one of the other and which do not cut the straight b, while every semi-straight passing through A and situated in the angle formed by a_1 and a_2 meets b.''

''It is these two demi-straights, a_1 and a_2, which have received the name of *parallels.*'' They do not meet the straight b, but they serve as *limit* to the angle wherein are found the straights which meet b and to the angle wherein are found the straights which do not meet b.

I would signalize an elegant theory of what might be called the points at infinity of the Lobachevskian plane, and of which the laws are the same as those of the addition and the multiplication of real numbers. One may draw thence a very simple and very suggestive exposition of the non-Euclidean geometry.

The origin of our acquaintance with the theory of parallels is found in the theorems of Legendre which establish a necessary correlation between the sum of the angles of a triangle and the choice between the three geometries, Euclidean, Lobachevskian, Riemannean.

What rôle does the axiom of Archimedes

play in these theorems? This question interested Hilbert and under his inspiration M. Dehn has made it the subject of a thesis which I can not pass over in silence. The conclusions of Dehn show that without the axiom of Archimedes the theorems of Legendre are no longer true.

It is still true that if *one* triangle has the sum of its angles equal to (or greater than) (or less than) two right angles the same is true of all the others. It is still true that if this sum is less than two right angles, one can draw many parallels to a straight through one point. It is true that if it is greater than two right angles, the postulatum of Euclid is false, and that if it is equal to two right angles it is impossible that two straights always meet, but *the other theorems of Legendre are not true.*

There exists a plane geometry where, the sum of the angles of a triangle being greater than two right angles, one can draw to a straight through one point an infinity of parallels (so I call straights which do not meet); this is the *non-Legendrean* geometry.

A geometry exists where the sum of the angles is equal to two right angles, and where one can draw to a straight through one point an infinity of parallels. This is the *semi-Euclidean geometry.*

It will suffice to explain here what this latter is, the former being altogether analogous. For this it is necessary to return to what I have said of the non-Archimedean geometry. I have explained how the non-Archimedean plane is deduced from the ordinary plane by the adjunction of new points; how for deducing a non-Archimedean straight D_1 from the ordinary straight D_0, it is necessary to add to it:

1. On the one hand, an infinity of new points between every two demi-straights S' and S'' of which the totality forms D_0.

2. On the other hand, an infinity of new points to the right of all the ordinary points of D_0, and an infinity of new points to the left of all the ordinary points of D_0. Well, retain the new points of the first sort, that is to say, those which are at a finite distance, and suppress the new points of the second sort, that is to say, those which are at an infinite distance.

Then let D be any straight and A any point; then there will be an infinity of straights passing through A and which do not meet D, those, namely, which would have met it in one of the new points of the second sort, if these points had not been suppressed. However, all the theorems of Euclid remain and every rotation or every translation will transform into itself the non-Archimedean plane so mutilated.

It seems that here is a contradiction with the results of the article just cited: 'Ueber eine neue Begründung. * * *'

If, as Hilbert has shown, the geometry of Lobachevski can be deduced from his postulate without the intervention of the axiom of Archimedes, how can there be a geometry semi-Euclidean, that is to say a geometry where the theorems of Euclid accord with the postulate of Lobachevski?

It seems that this difficulty springs from this, that the enunciation of the postulate is not the same in the two cases.

Dehn assumes that through a point one can draw an infinity of straights which do not meet a given straight, and an infinity of straights which meet it.

The first form an ensemble E_1, the others form an ensemble E_2. Hilbert supposes, in addition, that there exists a limiting straight which appertains to the ensemble E_1, and such that every straight comprised between this limit straight and a straight of E_2 appertains likewise to E_2. It is this limit straight which Hilbert considers as the *parallel properly so called.*

In the geometry of Dehn this parallel

properly so called does not exist. Here would be an interesting question to examine closely. Is it possible to construct a non-Archimedean geometry where this parallel properly so called exists and to which are applicable the results of Hilbert?

An analogous question is treated in another article by Hilbert, 'Ueber die Gleichheit der Basiswinkel im gleichschenkligen Dreieck.'

In the ordinary plane geometry, the plane is symmetric, which expresses itself by the equality of the angles at the base of the isosceles triangle.

One should make this *symmetry of the plane* figure in the list of metric axioms. In all the geometries more or less strange of which we have hitherto spoken, in those at least where one admits the metric axioms, in the metric non-Archimedean geometry, in the new geometries of Dehn, in those which have made the subject of the memoir 'Ueber eine neue Begründung * * *' this symmetry of the plane is always assumed. Is it a consequence of the other metric axioms? Yes, as Hilbert shows, if one admits the axiom of Archimedes. No, in the contrary case.

There are non-Archimedean geometries where all the metric axioms are true, with the exception of this of the symmetry of the plane. Here is an example:

The non-Archimedean numbers previously defined may be infinite or finite or infinitesimal; but an angle will be always finite or infinitesimal because of the relation

$$\cos^2\varphi + \sin^2\varphi = 1.$$

An angle may, therefore, always be put under the form $\theta + \tau$, θ being an ordinary real number and τ a non-Archimedean infinitesimal.

We define then the rectangular coordinates of a point, the straights and the translations in the ordinary manner and define rotation in the following manner. Let α, β be the coordinates of the center of rotation; $\theta + \tau$ the angle of rotation; x, y the coordinates of any point before the rotation; x', y' its coordinates after the rotation; one will have

$$(x' - \alpha) + i(y' - \beta) = e^{(i\theta + \tau + i\tau)}[(x - \alpha) + i(y - \beta)].$$

Consider the group formed by the rotations about the origin. This group will not be permutable for the transformation which changes y into $-y$, nor for any transformation which retains the origin, which changes straights into straights and of which the square reduces to the identical transformation. *The plane is, therefore, not symmetric.*

All the other metric axioms subsist, however, as does also the postulatum of Euclid and even a new axiom which Hilbert calls *Axiom der Nachbarschaft* and which he states thus:

"Given any sect AB, one can always find a triangle in the interior of which can be found no sect congruent to AB."

This results easily from the equation of the circle. The equation of a circle of radius ρ and center α, β is in fact:

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2 e^{2\tau};$$

$$\frac{y - \beta}{x - \alpha} = \tan(\theta + \tau).$$

In return, it is not true that the angles at the base of an isosceles triangle are equal; it is not true that in a triangle one side is less than the sum of the other two; finally the theorem of Pythagoras on the square of the hypothenuse is not true.

"It is for this reason that this geometry is called *non-Pythagorean.* [I may interpolate here that Poincaré is in error in saying Hilbert shows that the equality of the basal angles can be proved from the other metric axioms if one admits the axiom of Archimedes. In addition the new Axiom der Nachbarschaft is used.]

"I come to speak now of an article entitled 'Ueber die gerade Linie als kürzeste Verbindung zweier Punkte' that I can not separate from a thesis on the same subject, written by Hamel under the inspiration of Hilbert. Here we are less far from home; not only is there no question of renouncing the axiom of Archimedes, but we encounter only analytic functions which may be differentiated and integrated.

"Suppose that one has defined straights in the ordinary fashion and that one admits the projective axioms, those of order and the theorems of Desargues and Pascal. Define now the length of an arc of any curve; it is not necessary to choose this definition so as to satisfy the metric axioms, that is to say so as to render possible the movement of a rigid figure.

"Is it possible so to make this choice that the straight line shall be the shortest path from one point to an other?

"The definition of the straight is not changed, but that of the circle is in a very large degree arbitrary; it is only necessary that all the circles which have their center on a straight and which pass through a point of this straight have at this point the same tangent. The problem permits an infinity of solutions.

"Minkowski, for an arithmetic purpose, has developed one of them where all the circles are curves similar to each other in the ordinary sense of the word. Hilbert, from 1894, had given another of them which may be thus characterized: We consider a connected closed curve which will serve as fundamental curve. Let D be a straight, M a point of this straight; all the circles which have their center on D and which pass through M have the same tangent T, and this tangent, when the point M describes the straight D, pivots around a fixed point which is the intersection of two tangents to C at the points where this curve is cut by the straight D.

"Finally Hamel has in his thesis given the general solution of the question, but this solution is too complicated to be expounded in few words.

"I arrive at an important memoir of Hilbert's which is entitled 'Grundlagen der Geometrie,' which bears, consequently, the same title as his 'Festschrift,' but where, however, he places himself at a point of view altogether different.

"In his 'Festschrift,' in fact, as one sees by the preceding analysis, the relations between the notion of space and the notion of group, as they result from the works of Lie, are left to one side or relegated to a secondary part. The general properties of groups do not appear in the list of his fundamental axioms.

"This is not so in the memoir of which we are about to speak. Hilbert assumes a plane about which he makes the following hypotheses:

"1. The points of this plane correspond one to one to the points of the ordinary plane or to a part of these points. Thus each point of the new plane has its correspondent in the ordinary plane; but there may be on the ordinary plane points which have no correspondent on the new plane.

"The new plane has, therefore, so to say, less points than the ordinary plane, which is the contrary of that which happened for the non-Archimedean plane. The points of the ordinary plane which have correspondents on the new plane are called Bildpunkte. The ensemble of Bildpunkte forms on the ordinary plane a region which Hilbert assumes continuous and connected in such fashion that around each point of this region one can describe a circle of radius sufficiently small to be contained in this region and that one can go from one point to the other of the region, in following a continuous curve and without going out from the region.

"2. The points of this new plane are susceptible of transformations called *movement* and which form a *group*.

"3. Among these movements, there are an infinity which leave fixed a certain point M and which are called rotations about M.

"The ensemble of the transformés of the same point A by all these rotations is called a circle. Every circle has an infinity of points.

"4. The group of movements forms a *closed system;* which means this: if there are an infinity of movements which change two points A_0 and B_0 the first into A_1 and B_1, the second into A_2 and B_2, $\cdots$ the nth into A_n and B_n; and if the point A_n tends towards A and the point B_n towards B when n increases indefinitely, there will also be in the group a movement which will exactly change A_0 into A and B_0 into B; and the same holds if in place of two points we consider three of them or only one.

"I have slightly abridged the statements, making them lose, it is true, a little of their precision, but without taking away anything essential. About these enunciations I have certain observations to make.

"The question in brief is to find all the groups of transformations of the plane into itself, or of a part of the plane into itself, these groups being subjected only to conditions in appearance very slightly restrictive. How, therefore, can one arrive at conclusions so precise?

"This results from the definition which Hilbert gives of movement. To be a movement, a transformation must satisfy many conditions; first it must be continuous and transform two points infinitely near into two points infinitely near; then it must be biuniform, that is to say, that every point of the plane must have one transformé and only one, and be the transformé of one point and of only one.

"By that are found to be excluded a very great number of groups; for example, the group of the projective transformations of the plane and the group of homothetics, that is to say, transformations which change every plane figure into a homothetic figure [a figure similar and similarly placed].

"Why? If we take, for example, the homothetic group we see that it contains degenerescent transformations, those where, the center of homothety moreover being any whatever, the ratio of homothety is nul or infinite. In these transformations the center of homothety has an infinity of transformés or is the transformé of an infinity of points. These degenerescent transformations, without which the group would not be a *closed system,* can not be excluded, nor any more can they be retained, since they do not satisfy the definition of movement.

"One may see in the same manner that a circle can not contain all the points of the plane, otherwise, among the rotations about the center of this circle, there would be one which would bring to the center a point of the plane, other than the center, so that the center would be the transformé of two points, of this point and of itself.

"That implies the existence of an invariant analogous to distance.

"One sees that the conditions are more restrictive than they seemed. In relation to the ideas of Lie, the progress realized is considerable. Lie supposed that his groups were defined by analytic equations.

"The hypotheses of Hilbert are much more general.

"Without doubt this is not yet entirely satisfactory, since if the *form* of the group is supposed any whatsoever, its *matter,* that is to say, the plane which undergoes the transformations, is still obliged to be a

Zahlenmannigfaltigkeit in the sense of Lie. This is, nevertheless, a step in advance, and besides Hilbert analyzes better than any one had done it before him the idea of Zahlenmannigfaltigkeit, and gives hints which may become the germ of an axiomatic theory of analysis situs.

''I can here only summarize the general march of Hilbert's ideas.

''He shows first that the points of a circle can be arranged in a determined circular order and that this order is not altered by rotations; then he shows that this order falls into the same *type of order* as the corresponding order of the ordinary circle, that is to say, into the type of the continuous. Thence he deduces this consequence that the circle is a continuous closed curve, because it must correspond point for point to the ordinary circle.

''One sees then that if a rotation does not displace one point of a circle, it will not displace any other point of this circle. Thence one can deduce that if a rotation does not displace one point different from its center, it will not displace any of the points of the plane and will reduce to identity. From this results finally that the group of rotations around a point M has the same structure as the group of ordinary rotations.

''One sees at the same time that there is no movement which leaves fixed two points of the plane, and that we can pass by rotations from one point of the plane to any other point whatsoever of the plane.

''All these demonstrations are extremely delicate; they require the repeated employment of the theorems of Cantor.

''This is to say that they are perforce very long and that the goal which one perceives immediately and which one thinks to touch can be attained only by long efforts.

''The greatest step is then accomplished; now we know that our group derives from certain subgroups, those of rotations, of which we know the structure, and this structure makes these subgroups fall into the category of Lie's continuous groups.

''We have, therefore, only a few difficulties still to vanquish, but Hilbert wishes first to define the straight and he has done it in a very original fashion.

''He rejects first the projective definitions of the straight which require considerations foreign to his premises. On the other hand, his geometry is a *plane geometry*.

''If we may use space of three dimensions, the theory of groups leads us naturally to a very simple definition of the straight, considered as axis of rotation; but here we can not use this, since we can not go out of the plane.

''Hilbert follows wholly another way. Let there be two points, A and B; define the middle of these points, that is to say the center of a rotation which changes A into B and B into A. Hilbert begins by demonstrating that two points have always a middle and have only one. It is here that comes in an hypothesis which above must have astonished the reader; we have supposed that the last axiom (which one states in an abridged fashion in saying that the group of movements is a closed system) is applicable not only if one envisages two points A_0 and B_0, but also if we consider three points. We have, therefore, made an hypothesis more restrictive than if we had limited ourselves to the consideration of two points A_0 and B_0. Was this restriction really necessary?

''It is in this part of the theory that it plays its rôle. We consider an infinity of points $B_1, B_2, \cdots B_n$, and the middles $M_1, M_2, \cdots M_n, \cdots$ of the sects $AB_1, AB_2, \cdots AB_n$; when n increases indefinitely, B_n tends toward B and M_n toward M and we make use of the hypothesis in question to show that M is the middle of AB. Had it been impossible to use it, we could have been sure of this only after having constructed a special pseudogeometry.

"As it is, two points A and B being given, Hilbert constructs the middle of the sect AB, then the middle of the two sects MA and MB and so on. He thus obtains an infinity of points which form an ensemble E; he considers the derivative of this ensemble E, that is to say the assemblage of the limiting-points of E, points such that in any interval containing one there are an infinity of points of E. He shows that this derivative is a continuous line, and it is this line that he calls the straight [die wahre Gerade].

"The fundamental principles of the ordinary Euclidean or non-Euclidean geometry may then be easily established and in particular the metric axioms.

"It is impossible not to be struck by the contrast between the point of view where Hilbert places himself here and that which he had adopted in his 'Festschrift.' In this 'Festschrift' the axioms of continuity occupy the last rank and the grand affair was to know what geometry became when one threw them aside. Here on the contrary it is continuity which is the point of departure and Hilbert is especially preoccupied to see what one gets from continuity alone, joined to the notion of the group.

"It remains for us to speak of a memoir entitled 'Flächen von konstanter Krümmung.'

"It is known that Beltrami has shown that there are in ordinary space surfaces which are the image of the Lobachevskian plane, namely the surfaces of constant negative curvature; we know what an impulse this discovery gave to the non-Euclidean geometry. But is it possible to represent the whole entire Lobachevskian plane on a surface of Beltrami without singular point?

"Hilbert demonstrates that it is not; he founds his proof on the following theorems relative to the Beltrami surfaces.

"A quadrilateral formed of asymptotic lines has its opposite sides equal.

"The surface of a polygon formed of asymptotic lines is proportional to the spherical excess and it is the same with the surface of a polygon formed of geodesic lines; only in the first case the spherical excess is positive, in the second case it is negative.

"The author shows then that on a Beltrami surface without singular point one can not trace a closed asymptotic line; that an asymptotic line can neither cut itself, nor cut another asymptotic line in more than one point. Every other hypothesis would lead to polygons of which the spherical excess would be nul. Thence it follows as a consequence that if such a surface corresponds point for point to the non-Euclidean plane, this correspondence must be biuniform. But then in evaluating the total surface starting from the area of the polygon formed of asymptotic lines or from the area of the geodesic polygon we find in the first case that this total surface is finite, in the second that it is infinite. This contradiction demonstrates the theorem enunciated.

"In that which concerns the surfaces of positive constant curvature, to which the geometry of Riemann corresponds, Hilbert demonstrates that aside from the sphere there is no other closed surface of this sort. In fact, if we consider a portion of surface of constant positive curvature, the maximum of the great radius of curvature can not be attained *in the interior* of this portion, but only on its contour. This is a proposition entirely analogous to a well-known theorem relative to the potential.

"It follows thence immediately that if the surface is closed, the maximum can be nowhere attained and the radius of curvature is constant. Thus we easily come back again to the sphere.

"After this analysis, all commentary is useless. One sees at how many different points of view Hilbert has placed himself, how profound is his analysis.

"His works mark an epoch and he seems entirely worthy of the Lobachevski prize. —Poincaré."

The 'Report on the works of Monsieur Barbarin, professor of higher mathematics at the Lyceum of Bordeaux, relative to the non-Euclidean geometry,' is by Professor Mansion, of Ghent, as follows:

"I. *List of the Works of M. Barbarin.* —M. Barbarin has published, from 1898 to 1902, the following memoirs and works relative to the non-Euclidean geometry.

"1. Géométrie générale des espaces (*Association française pour l'avancement des sciences.* Congrès de Nantes, 1898, pp. 111–132).

"2. Propriétés angulaires des cercles focaux dans les coniques (*Ibid.*, 1898, pp. 132–139).

"3. Constructions spheriques à la règle et au compas (*Mathesis*, 1899, pp. 57–60; 81–85).

"4. Etudes de géométrie analytique non-euclidienne (*Mémoires couronnés et autres Mémoires* publiés par l'Académie royale de Belgique, 1900, t. LX., 167 pp. in 8°. This memoir was presented to the Royal Academy of Belgium, the fourth of December, 1897).

"5. Le cinquième livre de la Métagéométrie (*Mathesis*, 1901, pp. 177–191).

"6. Les cosegments et les volumes en géométrie non euclidienne (Extrait des mémoires de Bordeaux, 1901; 20 pp. in 8°).

"7. La géométrie non euclidienne. Paris, Naud, February, 1902 (collection *scientia*, 79 pp. in 12°).

"8. Bilatères et trilatères en Métagéométrie (Mathesis, 1902, pp. 187–193).

"9. Polygones réguliers sphériques et non euclidiens (*Le Matematiche pure ed applicate*, 1902, t. II., pp. 137–145).

"We will now analyse these works, classing the results found by M. Barbarin under three heads:

"*Elementary geometry, conics and quadrics, infinitesimal geometry.*

"II. *Elementary geometry.*—In his little book entitled 'The non-Euclidean geometry' (List No. 7), M. Barbarin expounds the first principles of the geometry, especially after Saccheri, Bolyai and Lobachevski and, among the moderns, DeTilly, Gérard, Mansion. But, besides, he makes known, whether in this little book or in divers special notes, results which are his own.

"1. *Bilaterals and trilaterals* (List No. 8).—The author proves in an elementary manner, *without recourse to analysis*, that the locus of points equidistant from two straights is a straight; that the bisectors, the medians and the altitudes of a trilateral are copunctal (meet in the same point) [real at a finite or infinite distance, or ideal], even if the vertices of the trilateral are all or in part reals at infinity or ideals.

"He deduces from the theorem on the three altitudes a novel construction of the normal common to two Lobachevskian straights which only meet at an ideal point.

"2. *Fundamental constructions* (List No. 3, No. 4, § I., pp. 5–14; No. 7, pp. 46–49).—M. Barbarin gives the means of constructing, with the ruler and the compasses, a right-angled triangle or a trirectangular quadrilateral given two elements and thence deduces all the fundamental constructions of the non-Euclidean geometry.

"He depends, in these constructions, on the old theorems of Lobachevski and Bolyai and on three new theorems which seem to have escaped these illustrious geometers. Here they are, in Lobachevskian geometry:

"Let *ABCD* be a quadrilateral trirectangular in B, C, D, and having, consequently, angle DAB acute; E — a point situated between D and A, F — a point situated between B and A, such that $CE = BA$, $CF = DA$.

"We have (1) $BF = DE$. (2) CE is asymptote (Lobachevskian parallel) to BA, CF to DA. (3) The perpendiculars let fall on CB, CD from points equidistant from C taken on CE, CF intersect on the diagonal CA.

"3. *Regular spheric and non-Euclidean polygons* (List No. 9.—M. Barbarin has found constructions simple and novel for the regular polygons of 3, 6, 5, 10, 15 sides, applicable at the same time in Euclidean geometry and in non-Euclidean geometry of Riemann and of Lobachevski, *for the sphere and for the plane.*

"4. *The fifth book of Metageometry* (List No. 4, ch. IV., pp. 94–99, No. 5, No. 7, pp. 37–41).—M. Barbarin calls fifth book of 'Metageometry' that which corresponds to the fifth book of the 'Elements of Legendre' or to the eleventh of Euclid.

"He makes an elementary exposition of it more complete than does any of his predecessors; here and there it could have been intuitive if he had depended more on the asymptotic property of Lobachevskian parallels. There is room to cite in this work the two following theorems: (1) That a right angle may be projected upon a plane into a right angle, it is necessary and sufficient that the projector of the vertex be normal to the plane and to one of the sides of the angle.

"(2) Two Riemannean straights not situated in the same plane have two common normals; if these normals are equal, the two straights are equidistant.

"Descriptive non-Euclidean geometry rests on the first proposition.

"From the second, it results that there exist, in Riemannean geometry, skew squares and rectangles having four right angles and surfaces equidistant from a straight with rectilinear generators.

"5. *Coordinates; geometry of n dimensions* (List No. 1, No. 4, § II., pp. 14–28, § IV., pp. 84–101).—In his 'Studies in non-Euclidean Analytic Geometry,' M. Barbarin has been led to certain new developments of the theory of coordinates, and, consequently, to expound by the calculus, the fundamental properties of the straight and of the plane, of angles and of distances, of the circle and of the sphere.

"This is, in the main, under analytic form, the complement of his other studies on the 'Elements.'

"The memoir of pure analysis, entitled 'General Geometry of Spaces' is a generalization of the formulas of Euclidean or non-Euclidean geometry of three dimensions relatively to the straight, to the plane, to the triangle and to the trihedrals and the trigonometric relations relative to them, when one considers a variety of n dimensions. The author shows, in particular, that for such varieties there exists also a limit-case that we may call *Euclidean geometry of n dimensions.*

"III. *Conics and quadrics* (List No. 2, No. 4; the essential part of 2 is reproduced in 4). The 'Studies in non-Euclidean Analytic Geometry' constitute M. Barbarin's largest work. It is devoted, for the major part (§ III., pp. 29–84; § V., pp. 101–139), to a classification of conics and quadrics more complete than that of his predecessors, without having any recourse to the Cayleyan geometry.

"A. *Conics.*—The author first reduces to its most simple forms the general equation of the second degree or rather a ternary quadratic form.

"In Riemannean geometry he finds only two kinds of curves: *imaginary conic* and *ellipse,* the latter having the *circle* as variety.

"The Lobachevskian geometry is much

richer in curves of the second degree. The curves with real center are the *ellipse* (real, semi-real, ideal or imaginary) with the important varieties, *circle* and *hypercycle* (= equidistant from the straight). The curves denuded of center, even at infinity are the parabolas (elliptic, veritable, hyperbolic). The curves with center situated at infinity are the *oriconics* (oriellipse, with the variety oricycle of Lobachevski, orihyperbola).

"M. Barbarin generates the lines of the second degree by homography and by movements of linkages; he investigates their foci, their focal circles and their directrices; finally their reciprocal curves. Two reciprocal curves of the second degree are such that each is the locus of the center (real or ideal) of the tangents of the other. The properties of these curves are a consequence of the principle of duality, which is, so to say, evident in non-Euclidean geometry.

"The author then studies the plane sections of a cylinder or of a cone of the second degree (that is to say, having for plane directrix a curve of the second degree); he finds again in this way all the varieties of curves of the second degree, which are, therefore, truly conics. He extends to non-Euclidean conics the most celebrated theorems relative to Euclidean conics and, in particular, those of Dandelin. He obtains in a manner more systematic still all the curves of the second degree, Riemannean, Euclidean, Lobachevskian, by cutting the cone of the second degree by a concentric sphere, the common center being real, at finite or infinite distance, or ideal.

"One again finds conics in cutting by a plane the straight equidistant surface (tube of revolution with rectilinear axis, or hypercycloide of revolution). In Riemannean geometry there is a case where one finds as section two straights equidistant from the axis, but not coplanar with the axis (compare above, II., 5): these straights are the *helices* of this surface.

"B. *Quadrics.*—The reduction of the general equation of the surfaces of the second degree is deduced from the discussion of the equation in s of the nth degree.

"In Riemannean space, we find two principal species—*ellipsoid* (with the varieties ellipsoid of revolution, tube sphere), *pipe-hyperboloid* (with the varieties cone, hyperboloid of revolution or elliptic tube, two planes).

"In Lobachevskian space we find first the species *ellipsoid* (with three unequal real axes, semi-real with two real axes, with one real axis or imaginary); the *first hyperboloid* (with one nappe real, with two nappes real, with one nappe ideal); the *second hyperboloid* (with two nappes real or ideal). The varieties or limits of these three species are very numerous. All these surfaces have a center and three principal planes.

"The *paraboloids* (elliptic, semi-elliptic or hyperbolic) and their numerous varieties have no center and have two principal planes. They *cut* the sphere of infinite radius.

"At the limit, when they become *tangent* to this sphere, they are transformed into oriquadrics (oriellipsoid, orihyperboloid) and into their varieties.

"M. Barbarin has studied the rectilinear and circular sections of these surfaces, their focal spheres and their directrices.

"IV. *Infinitesimal geometry.* 1. *Measure of areas and of volumes* (List No. 4, pp. 164–167; No. 6; No. 7, pp. 50–59).—M. Barbarin in his little book, 'The non-Euclidean Geometry,' summarizes in some pages the results found by Lambert, Lobachevski, Simon, etc.; but he calls attention also (List No. 6) to an original idea of which he is the author. He has remarked that the volume of a frustum of a non-Euclidean cone of revolution is proportional to the difference between the projection of the generatrix on the axis and this generatrix multiplied by the cosine of the angle which it makes with the axis if it meets it, or of the normal common to the generatrix and axis. He has thence deduced a *general formula for volumes decomposed into infinitesimal spindles of revolution.*

"By means of this formula he has been able to reach a number of known results; especially he has been able to make an advance in the very difficult question of the volume of the tetrahedron, which, as is known, has arrested Gauss, Bolyai and Lobachevski and all their successors.

"M. Barbarin finds an expression for the volume of the tetrahedron *where are introduced naturally the products of the edges by the corresponding dihedrals.* To achieve the solution of the question, it is requisite to find, under a finite form, certain functions relative to the faces which present themselves in the calculations under the form of integrals or complicated series.

"2. *Geodesic lines of tubes and pseudospheres* (List, No. 4, pp. 139–164).—In the last section of his 'Studies in non-Euclidean Analytic Geometry' the author has reached one of the most beautiful theorems of metageometry.

"It has been known, since Lobachevski and Bolyai, that the characteristic geometry of orispheres is Euclidean; since Beltrami, that of the Euclidean pseudosphere is Lobachevskian; finally it is evident that that of the sphere is Riemannean.

"The theorem of Barbarin (it is to be hoped that it will retain this name) comprises and generalizes in the most unexpected manner these particular propositions. Here it is in its most condensed form: *Each of the three spaces, Euclidean, Lobachevskian, Riemannean, contains surfaces of constant curvature of which the geodesic lines have the metric properties of the straights of the three spaces.*

"These surfaces are the spheres (characteristic geometry, Riemannean); the tubes or surfaces equidistant from a straight, it being possible for the distance to be infinite, which gives the orispheres (characteristic geometry, Euclidean); finally the pseudospheres, that is to say the surfaces of revolution which have for meridians a tractrix or line of equal tangents (characteristic geometry, Lobachevskian).

"The property of the surfaces equidistant from a straight is almost evident and has been found also by Whitehead; but the existence of Lobachevskian tractrices and pseudospheres and above all of Riemannean and the properties of their geodesics were not suspected before M. Barbarin.

"The curvature of the pseudospheres is negative in Riemannean space as in Euclidean space; it is negative, nul or positive in Lobachevskian space.

"V. *Résumé and Conclusion.*—Non-Euclidean geometry owes to M. Barbarin (1) fundamental properties of the plane trirectangular quadrilateral; (2) the discovery of Riemannean equidistant straights; (3) the complete classification of non-Euclidean conics and quadrics; (4) the most intuitive formula that we know for the determination of volumes, with a remarkable application to the tetrahedron; (5) finally and above all the beautiful general theorem cited above on the geodesics of tubes and pseudospheres, in the three geometries.

"All these results have been obtained by the direct study of the figures without borrowing anything from the Cayleyan geometry.

"If Lobachevski should come back to the world, he would recognize in M. Barbarin a worthy continuer of his work, inspired with his ideas and his method.

"We believe, therefore, that the Physico-mathematic Society of Kazan may legitimately decree to him the Lobachevski prize. —P. Mansion."

To Professor Hilbert I am particularly, personally indebted. My 'Rational Geometry' is an attempt to give every teacher, every scholar the benefit of coming after him, the priceless advantage of living since the outpouring of his genius.

Monsieur Barbarin has honored me with his genial friendship.

I suppress the temptation to institute comparisons or discuss a decision.

GEORGE BRUCE HALSTED,
Membre d'honneur du Comité Lobatchefsky.

THE TROPHOBLAST: A REJOINDER.

THE name *Trophoblast* was used for the first time by me in the meeting of the Anatomical Congress at Würzburg in 1888, and its earliest definition is found in the report of that meeting in Nos. 17 and 18 of the *Anatomischer Anzeiger*, Bd. III. We there read, concerning a very early stage of the hedgehog (p. 510):

"Die äussere Wand der Keimblase ist verdickt (drei bis vierschichtig) und besitzt wabige Lacunen. Für diese äussere (epiblastische) Schicht sei der Name *Trophoblast* gewählt."

In a footnote we find in addition (p. 511):

"Es ist meiner Ansicht nach zweckmässig, sich bei der Säugethier-embryologie diesen Namen zu wählen, um damit den nicht zum Aufbau des Embryos verwendet werdenden Epiblast anzudeuten * * *."

It is evident from the citations here given that the names outer epiblastic wall of the mammalian blastocyst and trophoblast are synonyms. Later researches have been directed towards the question how in other mammals than the hedgehog the separation between the epiblast of the embryonic shield, *i. e.*, the formative epiblast and the trophoblast, comes about.

In the same Bd. III. of the *Anatomischer Anzeiger*, on p. 907, mention is again made of the hedgehog's 'geschlossene Trophoblastblase (wie ich den primären Epiblast, von dem sich durch Abspaltung der Epiblast des Fruchthofes nach innen abhebt, zu benennen vorschlug).'

Again, in the article on the placentation of *Erinaceus* in Vol. 30, Pt. 3 (1889), of the *Quart. Journ. of Microsc. Science*, where the definition was reproduced, it is insisted upon (p. 298) that "the use of the name trophoblast will render unnecessary such circumlocutory expressions as 'outer epiblastic layer of the blastocyst,' 'primitive exochorion,' etc." Further argumentation on p. 299, in which the allantoidean and the omphaloidean trophoblast is defined, leaves not the faintest doubt as to what the name *trophoblast* has originally stood for.

Five years later (1894), in an article, 'Spolia nemoris,' which appeared in Vol. 36 of the *Quart. Journ. of Micr. Science*, I again insisted (p. 111) that 'new and valid reasons are thus accumulated for designating the outer layer of precociously segregated epiblast cells that form the wall of this vesicle [the early mammalian blastocyst] by a separate name, [for which] I have proposed the name of trophoblast.' Somewhat further is added (p. 112): 'in *Tupaja* and *Tarsius* portions of the trophoblast undergo very active proliferating processes preparatory to the placentary fixation of the blastocyst, whereas in my former papers I have described the same activity for *Erinaceus* and *Sorex*.'

Finally, in 1895 (*Verhandl. der Kon. Akad. v. Wetenschappen te Amsterdam*, Vol. IV., No. 5, p. 18), I reaffirm that: