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*SOME RECENT APPLICATIONS OF FUNCTION THEORY TO PHYSICAL PROBLEMS.**

IT has seemed appropriate that the address of the retiring chairman should draw attention to some of the most recent developments in those sciences which it is the object of this Section of the Association to promote, especially to some problems that seem to be making but slow headway, and to others that are at a standstill for want of appropriate modes of mathematical expression.

In selecting a particular group of problems I have been guided by the thought that there is one field of work which touches the domain of every member of this Section, whether his or her immediate interests lie in abstract mathematics, in physical mathematics or in astronomy. I mean the great field of the theory of functions of a complex variable.

The physicist or astronomer who wishes to understand the true nature of any function which he deals with must study its behavior on the complex plane, its zeros, its poles, its singularities and perhaps its Riemann surface. Moreover, in dealing with such important questions as stability

* Address by the retiring Vice-President and Chairman of Section A—Mathematics and Astronomy—of the American Association for the Advancement of Science, Pittsburgh meeting, June 28 to July 3, 1902.

and instability it is necessary to examine the region of convergence of the infinite series which so often present themselves; and this cannot be done with certainty without the methods of function theory.

In such cases we use the function theory to test the character of the solutions already obtained, and to find out the regions within which they are applicable; but in the discovery of solutions of new physical problems the methods of general function theory have seldom been used. It is chiefly of its use as an instrument of discovery that I wish to speak to-day.

It has long been known that the theory of functions of a complex variable is useful in treating the numerous physical problems whose solution can be made to depend on Laplace's equation in two dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This equation presents itself in the theory of the two-dimensional potential, and in problems relating to the steady flow of heat, of electricity and of incompressible fluids.

The essential feature of the method in question is to take an arbitrary function of the complex variable, and to express this function in the form

$$f(x + iy) = \phi(x, y) + i\psi(x, y),$$

in which φ and ψ are real functions of two real variables, x and y.

The functions φ and ψ are then said to be conjugate to each other, and are in all cases solutions of Laplace's equation, whatever be the assumed function f.

Moreover the two families of curves

$$\begin{aligned} \phi(x, y) &= C_1, \\ \psi(x, y) &= C_2 \end{aligned}$$

(in which C_1 and C_2 are arbitrary constant parameters) cut each other at right angles. The curves of one system may be taken as equipotential lines, and those of the other system will then be lines of force, or lines of flow. The physical boundary of the region must be some one of the lines of either set.

Some interesting applications of this method to tidal theory have recently been made by Dr. Rollin A. Harris in his 'Manual of the Tides,' published by the U. S. Coast and Geodetic Survey.* I would mention especially his use of an elliptic function as the transforming function in the form

$$x + iy = sn(\phi + i\psi).$$

The two sets of orthogonal curves drawn by him may be seen in the *Annals of Mathematics,* Vol. IV., page 83. By imagining thin walls erected along certain of the stream lines, we see, for instance, the nature of the flow around an island lying between two capes.

The direct problem of determining a solution of Laplace's equation that shall be constant at all points of a boundary previously assigned is usually very difficult. It is a particular case of what is commonly known as the Problem of Dirichlet. Before stating this problem it is convenient to define a harmonic function. Any real function $u(x, y, z)$ which satisfies Laplace's equation, and which, together with its derivatives of the first two orders, is one-valued and continuous within a certain region, is said to be harmonic within that region. Dirichlet's problem may then be stated as follows:

To find a function $u(x, y, z)$ which shall be harmonic within an assigned region, T, and which shall take assigned values at points on the boundary surface S.

This problem has long been one of the meeting grounds of mathematicians and physicists. Some important mathematical theories have received their starting point from this and similar 'boundary-value problems.'

* Part IV., A, pp. 574–82.

In proving that a solution always exists, Dirichlet began by assuming as self-evident that among all the functions which satisfy the assigned boundary conditions, there is a certain function, u, for which the integral

$$\int\int\int\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]dxdydz,$$

taken throughout the region T, is a minimum. This assumption is usually called 'Dirichlet's principle.' If this principle be granted it can be shown by the calculus of variations that the function u satisfies Laplace's equation; and it is easy to prove by Green's theorem that there is no other solution.

It was first pointed out by Weierstrass that this assumption is not allowable. If only a finite number of quantities present themselves we can assume that there is a smallest one among them. But among an indefinite number of quantities in any assigned group a smallest one does not necessarily exist. Consider for instance those rational numbers which decrease towards the square root of 2 as a limit; there is no smallest among them.

This led mathematicians to seek for other proofs of the existence theorem; and many interesting developments in function theory have been the result. Very recently Hilbert has reexamined Dirichlet's assumption, and has succeeded in demonstrating it, so that it is once more available as a starting point for the existence theorem.

When the boundary of the region is rectangular, circular, spherical, cylindrical, conical or ellipsoidal, the appropriate harmonic functions will be found in such works as Byerly's 'Fourier Series and Spherical Harmonics.'

I may mention here a new method of obtaining solutions of Laplace's three-dimensional equation used by Dr. Harris, and applied to tidal problems.* He uses the more general complex variable containing two imaginary units i and j. An arbitrary function of the form

$$\phi(ax+iby+jcz)$$

is a solution of Laplace's equation, provided $i^2=j^2=-1$, and $a^2=b^2+c^2$. When this function is expanded, the real part, and the coefficients of i, of j and of ij, are all separate solutions of the differential equation. A great number of solutions of this and similar equations can be obtained by this method. It is to be hoped that Dr. Harris may have time to develop it further.

In order to lead up to some recent applications of function theory I wish to speak especially of another method of solving Dirichlet's problem, namely by the use of Green's function.

Green's function is defined as follows for a given closed boundary S and a given pole P_1, within the bounded region T.

Let (x, y, z) be the current point within the region, and let (x_1, y_1, z_1) be the pole. Then $G^{x,y,z}_{x_1,y_1,z_1}$ is to vanish at every point of the boundary S, and is to be harmonic within the region T except at the pole (x_1, y_1, z_1), where it is to become infinite as $1/r$, where r is the distance of the current point (x, y, z) from (x_1, y_1, z_1).

There is always one and only one Green's function for a given boundary and pole. The determination of the form of this function G furnishes a solution of Dirichlet's problem; for it has the property that the surface integral

$$\int\int V\frac{dG}{dn}dS,$$

taken over the boundary of S, has the value $4\pi V(x_1, y_1, z_1)$, where V is any function harmonic within S, and dG/dn is the normal derivative of Green's function. Hence the value of V at any point

* 'Manual of Tides,' Part IV., A, pp. 584, 597.

(x_1, y_1, z_1) within the boundary is expressible in terms of its surface values and the normal derivative of G. Thus the solution of Dirichlet's problem is reduced to a problem in integration when Green's function is known.

Some recent advances have been made in determining Green's function for certain boundaries. To make them clearer I shall begin with the simple problem of finding Green's function for a region bounded by two planes at right angles and extending to infinity. Here Lord Kelvin's method of images is directly applicable. Let P_2 be the image of the pole P_1 taken with regard to the first plane. Let P_3 be the image of P_2 with regard to the second plane; and P_4 the image of P_3 as to the first plane. Then the image of P_4 as to the second plane brings us back to the first point, P_1. These four poles form a closed system, and there is only one pole in the given region. The required Green's function is

$$\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}$$

in terms of the distance of the current point (x, y, z) from the four poles; for this function, being a potential function, satisfies Laplace's equation; it also vanishes on the bounding planes by symmetry, and at infinity; moreover it becomes infinite as $1/r_1$ at the pole P_1, and is infinite nowhere else within the bounded region.

It may be observed that a direct physical interpretation of Green's function is illustrated by this problem. It is evidently the combined potential due to a positive unit of electricity placed at P_1 and to the induced charge on the bounding planes made conducting and maintained at zero potential; for this distribution realizes the boundary conditions. Hence the induced charge due to P_1 is equivalent in effect to three-point charges, namely, a positive unit at P_3, and negative units at P_2 and P_4.

Next consider the problem in which the angle of the planes is not an aliquot part of π. The simplest case is when this angle is $2\pi/3$. Performing the successive reflections as before, it is found that there are five reflections before the image comes back to P_1. There are then six poles, of which two are situated in the given region. The function

$$\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}+\frac{1}{r_5}-\frac{1}{r_6}$$

satisfies all the conditions except that of having only one pole within the region. It is thus not the required Green's function; and Lord Kelvin's method of images does not furnish a solution.

This method fails in two large classes of problems: (1) When the successive images (or poles) do not form a closed system; (2) when more than one of these poles lie within the assigned region.

By the conception of a Riemann space, Dr. Sommerfield* has recently made the important advance of overcoming the difficulty arising from the presence of two poles within the region. He regards the whole region as undergoing successive reflection; and thus, in the problem last mentioned, the whole of space is filled twice over. He imagines a two-fold Riemann space having the intersection of the planes as a winding line, and one of the planes as a branch membrane. The appropriate coordinates are cylindrical (r, θ, z). The axis of z is the line of intersection, and the plane $z=o$ is the plane passed through the original pole P_1, perpendicular to the axis of z. The radius-vector r is the distance of the current point from the z-axis, and θ is the angle which r makes with one of the planes, taken as initial plane.

* *Proc. Lond. Math. Soc.*, 1897, 'Ueber verzweigte Potentiale im Raum.'

When any radius vector OP revolves about the axis of z, it remains in the first (or physical) space until $\theta=2\pi$. It then crosses the branch membrane and enters the second fold of the Riemann space. In the problem before us, a second revolution brings the radius vector into the first fold again. It is to be understood that each fold fills all space. Two underlying points have the same r and the same z, but their θ coordinates differ by 2π or some odd multiple of 2π. Two points whose vectorial angles differ by an even multiple of 2π are in the same fold.

The problem now is to find a Laplace's function which shall vanish on each plane and at infinity, and shall have only one pole in the original physical space between the planes.

Let (r_1, θ_1, z_1) be the coordinates of the assigned pole, and (r, θ, z) those of the current point. Then $1/R$ is a solution of Laplace's equation, where

$$R^2=(z-z_1)^2+r^2+r_1^2-2rr_1\cos(\theta-\theta_1).$$

Dr. Sommerfeld first replaces θ_1 by an arbitrary parameter α, and denotes the result by R'. He then multiplies $1/R'$ by an arbitrary function $f(\alpha)$, and integrates with regard to α. The result is still a solution of Laplace's equation. By a proper choice of the function $f(\alpha)$, and of the range of integration, he obtains a function of (r, θ, z) satisfying all the conditions. He takes the two-valued function

$$f(\alpha)=\frac{e^{\frac{i\alpha}{2}}}{e^{\frac{i\alpha}{2}}-e^{\frac{i\theta_1}{2}}},$$

and puts

$$u_1=\frac{1}{4\pi}\int\frac{1}{R'}\frac{e^{\frac{i\alpha}{2}}}{e^{\frac{i\alpha}{2}}-e^{\frac{i\theta_1}{2}}}d\alpha,$$

then regards α as a complex number, and performs the integration in the α-plane around a contour enclosing the point $\alpha=\theta_1$ and excluding the other points where the integrand becomes infinite. The function u_1 thus obtained becomes infinite at the pole (r_1, θ_1, z_1) but does not fulfill the condition of vanishing on the two planes. Next he forms a similar function u_2 for the pole P_2 situated at (r_2, θ_2, z_2), and so on. The required Green's function is

$$u=u_1-u_2+u_3-u_4+u_5-u_6.$$

The poles of u_1 and u_4 would, under ordinary circumstances, both lie in the given region, but the pole of u_4 is given such a vectorial angle as to bring it into the second fold of the Riemann space. The function u has then only one pole for the physical region defined by $0<\theta<2\pi$.

Moreover, u vanishes for points on the two planes, and fulfills all the other conditions for Green's function.

Thus we see how a function, which would be two-valued and bi-polar if restricted to the given physical region, becomes single valued and uni-polar in the Riemann space. We may say that the second fold of this space is a refuge for the second value and the second pole. Care has to be taken to use the proper values for θ when the indicated operations are being performed. The difficulties of the problem are thus reduced to those of the integral calculus.

In the more general case in which the angle of the planes is $n\pi/m$, there are $2m$ poles in the circuit (one in each angle π/m), of which n are in the given region. The Riemann space is then n-fold.

Sommerfeld has worked out at length the very interesting case in which the angle between the planes is 2π. The region is then bounded by the surfaces $z=\pm\infty$, $r=\infty$, $\theta=0$, $\theta=2\pi$; the last two being the two faces of an infinite half plane with a straight edge. The assigned pole and its image are both in the given region; hence the corresponding Riemann space is two-fold; and the required solution is

$$\bar{u}=u(\theta_1)-u(-\theta_1)$$

where u is of the same form as u_1 written above.

By inversion with regard to different centers, various other problems are reduced to this one; for instance, the infinite plane with a circular aperture, the circular disc, and the spherical segment.

With regard to the uniqueness of the solution, Dr. Sommerfeld has proved by a remarkable use of function-theory methods that a function satisfying the conditions already laid down for Green's function is uniquely determined in a Riemann space.

I next speak of some recent advances in the solution of an equation more general than Laplace's, namely, the differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+k^2u=0,$$

which plays such an important part in the treatment of vibrating systems of various kinds; and I may introduce them by a quotation from Pockels' treatise on this equation: "Those solutions of our differential equation, which in accordance with their physical significance are regarded as single-valued within certain bounded regions, would by analytical continuation over the boundary in general become multiform. Therefore, from both a mathematical and a physical standpoint, multiform functions are important, and it is very desirable that the properties of such functions, their winding points and singularities, their behavior on Riemann surfaces, etc., should be systematically investigated—in short, all the function-theory questions which were handled in the theory of the Newtonian and logarithmic potential. * * * "Similarly as we have treated of solutions that are single-valued in the whole plane, it would be of interest to seek functions which are single-valued on a closed Riemann surface, or in an analogous three dimensional region, more especially those functions which are everywhere finite and continuous, namely the so-called 'principal solutions,' within the region in question. Finally there is the further investigation of the essential singularities and the natural boundaries which the functions satisfying this equation may present. * * * Investigations regarding these questions have not yet been made, more especially the integration of our equation for a closed manifold has hardly been touched. In this direction of inquiry without doubt a wide and rich field offers itself."

These words were written in 1890; and in 1897 appeared Professor Sommerfeld's paper on multiform potentials of which I have given some account above. He and his pupil, Dr. Carslaw, have also attacked the multiform solutions of the more general equation to which Pockels refers.*

The first problem that presents itself is to find a solution that has no pole, and is multiform with period $2n\pi$, in the ordinary sense, but on a certain n-sheeted Riemann surface is uniform. The case $n=2$ solves the following well-known physical problem:

Plane waves of sound, light or electricity are incident on a thin infinite half plane bounded by a straight edge, to find the resulting diffraction of the waves.

This problem had previously been mentioned by Lord Rayleigh in the article on Wave Theory in the Encyclopædia Britannica in the following terms:

"The full solution of problems concerning the mode of action of a screen is scarcely to be expected. Even in the simple case of sound where we know what we have to deal with the mathematical difficulties are formidable, and we are not able to solve such an apparently elementary question as the transmission of sound past a rigid infi-

* *Proc. Lond. Math. Soc.*, 1898; *Zeitschrift*, 1901; *Proc. Edin. Math. Soc.*, 1901.

nitely thin plane screen bounded by a straight edge or perforated with a circular aperture.''

Again the same author says in his work on the 'Theory of Sound':*

''The diffraction of sound is a subject which has attracted but little attention either from mathematicians or experimentalists. Although the general character of the phenomena is well understood, and therefore no very striking discoveries are to be expected, the exact theoretical solution of a few of the simpler problems, which the subject presents, would be interesting.''

Accordingly the recent solutions of Sommerfeld and Carslaw are very welcome to mathematicians and physicists. A very brief sketch of the principle of the method may here be given.

Let the waves come from the direction $\theta = \theta'$, and be incident on the plane $\theta = 0$. In the (x, y) plane, or in the (r, θ) plane, the origin will be regarded as a winding point, and the line $\theta = \pi + \theta'$ a branch line. Start with the simplest solution of our differential equation, namely, that for undisturbed plane waves in infinite space,

$$u = e^{ikr\,(\cos\theta - \theta')};$$

replace θ' by α, multiply by the same two-valued function of α as before, and integrate around the point $\alpha = \theta'$ in the complex α-plane. The result of the integration is a multiform solution of period 4π. The solution of the physical problem is obtained by adding the multiform solution for waves coming from the direction θ' to that for the direction $-\theta'$. There is, of course, considerable difficulty in performing the indicated operations, but this does not diminish the theoretical value of the solution, as the difficulties belong only to the integral calculus.

* 'Theory of Sound,' Vol. II., p. 141.

The next problem in order is that of waves issuing from a point-source against the half-plane, either in two or in three dimensions.

In the latter case we start with the undisturbed solution in infinite space

$$u = \frac{e^{ikR}}{R}$$

and treat this function as we treated 1/R in the potential problem. We put poles at (r', θ', o) and $(r', -\theta', o)$, and take the physical space as defined by $0 < \theta < 2\pi$.

It will be found that the function

$$\bar{u} = u(\theta') + u(-\theta')$$

satisfies all the conditions in the assigned physical space.

In the corresponding two-dimensional problem, the starting point is the undisturbed solution

$$u = Y_0(kr),$$

where Y_0 is the Neumann function.

The same method is applicable to problems in the flow of heat, in which the equation

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}$$

is to be satisfied. The starting point is the solution for a point-source in an infinite solid

$$u = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(\frac{-R}{4kt}\right).$$

In his recent paper published in the *Zeitschrift*, Sommerfeld has extended the method so as to apply to the problem of Röntgen rays encountering an obstacle represented by the same half-plane. He obtains a multiform solution of Maxwell's equations, and adapts it to the physical conditions, comparing the results with experimental data.

The induced currents flowing in an infinite half plane have been studied by Mr. Jeans* by the multiform method, using a

* *Proc. Lond. Math. Soc.*, 1899.

Riemann space with a single winding line.

The next advance was to solve a problem in multiform potentials in a Riemann space with two winding lines. Such a case presents itself in finding Green's function for an infinite plane with an infinitely long strip cut out. Sommerfeld has treated this problem by the use of the bipolar coordinate system

$$\rho = \log \frac{r_1}{r_2},$$
$$\phi = \theta_1 - \theta_2.$$

This is the system used so skillfully by Maxwell in which the curves $\rho = C_1$, $\varphi = C_2$ form two orthogonal families of circles (or cylinders). The Riemann space will have the straight lines corresponding to $\rho = \pm\infty$ for winding lines, and the plane $\varphi = 0$ for branch membrane.

The work of obtaining solutions of our differential equations on other Riemann surfaces or spaces has yet to be done. The difficulty lies in finding an appropriate system of coordinates. This is an attractive field and seems worthy of the attention of the best pure mathematicians.

It is interesting to note that the idea of obtaining a new solution by integrating an old solution in the complex plane with regard to a parameter seems to have occurred independently to a Scotch mathematician (J. Dougal, *Proceedings Edinburgh Math. Soc.*, 1901). For instance he regards the Bessel function $J_n(kr)$ as a function of its order n, and integrates with regard to n. The Legendrian and other functions may be treated in the same way. New functions are thus obtained that satisfy various boundary conditions.

All that I have said illustrates the need there is for new forms of functional relationship. The more new functions we can invent the better; that is to say, functions with new and varied characteristic properties. We look to general function theory to supply them. One never knows how soon they may find suitable use in some field of pure or physical mathematics. I said at the beginning that a number of physical problems are at a standstill for want of an appropriate mode of mathematical expression. In proof of this I may here quote the words of a few experts in different lines of work.

Lord Rayleigh says,* "When the fixed boundary of a membrane is neither straight nor circular, the problem of determining its vibrations presents difficulties which in general could not be overcome without the introduction of functions not hitherto discussed or tabulated. A partial exception must be made in favor of an elliptic boundary."

I may note here that Mathieu solved the problem of the elliptic membrane by transforming the differential equation to elliptic coordinates (ξ, η), so that one coordinate ξ would be constant on an elliptic boundary, and then satisfying the equation by means of a product function

$$u = \varphi(\xi) \cdot \psi(\eta),$$

making φ vanish on the boundary. This method might seem promising for other boundaries; but Michell has proved that the elliptic transformation is the only one that leads to an equation capable of being satisfied in the product form.†

Lord Rayleigh says in another place:‡ "The problem of a vibrating rectangular plate is one of great difficulty, and has for the most part resisted attack. * * * The case where two opposite edges are free while the other two are supported has been discussed by Voigt."§

In connection with air vibration he says: "The investigation of the conductivity for various kinds of channels is an important

* 'Theory of Sound,' Vol. 1, p. 343 (2d ed.).

† *Messenger of Mathematics*, 1890.

‡ 'Theory of Sound,' Vol. 1, p. 372.

§ *Göttingen Nachrichten*, 1893.

part of the theory of resonators, but in all except a very few cases the accurate solution of the problem is beyond the power of existing mathematics.''*

Professor E. L. Brown in his report on hydrodynamics presented to the Boston meeting says: ''No problem of discontinuous motion in three dimensions has yet been solved. The difficulty is one which can be easily appreciated. The theory of functions deals with a complex of the form $x+iy$ and this suits all problems in two dimensions. But little has been done with a vector in three dimensions. Perhaps the paper on Potentials by Sommerfeld in the *Proceedings of the London Mathematical Society* last year may have some bearing on the problem; it is in any case worth serious study. The subject of discontinuous motion was set for the Adams prize in 1895. A solution for a solid of revolution was asked for, and it was generally supposed that the circular disc would be the easiest to attempt. No solution was sent in. One prominent mathematician who has aided considerably in the development of hydrodynamics mentioned that he had worked for six months and had obtained absolutely nothing. A magnificent reception therefore awaits the first solution.''

Mr. Hayford writes (in SCIENCE, 1898): 'The most important tidal problem before us is that of determining the relation between the boundaries (bottoms and shores) and the modification produced by them on the tidal wave.'

Professor Webster, in his report on recent progress in electricity and magnetism, presented to the Boston meeting, says: ''The problem of electrical vibrations in a long spheroid is next to be attacked, and then perhaps on surfaces obtained by the revolution of the curves known as cyclides. The introduction of suitable curvilinear coordinates into the partial differential equations will lead us in the case of the spheroid to new linear differential equations, analogous to, but more complicated than, Lamé's, and will necessitate the investigation of new functions and developments in series.''

Dr. Webster also commends to the attention of pure mathematicians the various differential equations which are to be found in Heaviside's electrical papers; more especially the question of existence theorems.

I may mention here that Hilbert in a recent volume of the *Archiv** suggests the question of proving an existence theorem for the solution of any differential equation subject to assigned boundary conditions.

Even a partial treatment of any one of these problems might open up new relationships, and widen the intellectual horizon. It is a hopeful sign that several pure mathematicians are turning their attention to such questions. Speaking at the Chicago Mathematical Congress in 1893, Professors Klein and Webster deplored the growing separation of the pure and physical branches of mathematics, and pointed out the great loss that would result to each of the divergent branches. The recent increased attention to mathematical history has enforced this opinion. The influence of Klein, Poincaré, Weber and others has been helpful as a corrective, on the continent of Europe. The British Universities have steadfastly treated mathematical physics as an organic part of mathematical discipline. The same statement could not be made with regard to all of the American Universities; but there are many signs of improvement. With a true historical instinct, this Section of the Association, and its ally, the American Mathematical Society, have exerted their influence for an

* 'Theory of Sound,' Vol. 2, p. 175.

* *Archiv Math. und Phys.*, 1901, p. 229.

organic union of the entire mathematical field. On the whole, the indications are that the separation which was so deplored ten years ago is now being arrested.*

Besides the discovery of new functions a useful work might also be done in the tabulation of old ones. Our sister Association in England has set us a good example in this respect. The tables of elliptic integrals given by Legendre ought to be extended; and tables for the elliptic functions would be welcomed. The Neumann function needs tabulation, and several others might be mentioned. The familiar functions ought also to be tabulated on the complex plane. The labor could easily be divided up. I have myself made a beginning of this kind of work by computing the trigonometric and hyperbolic sine and cosine of $x+iy$ for values of x and y ranging separately from 0 to $\frac{1}{2}\pi$ at intervals of .1; it was published in Merriman and Woodward's 'Higher Mathematics,' 1896, and I have already had my reward in the fact that one electrical engineer has told me that he has used this complex table in the application of vector-theory to alternating currents. In connection with the chart already referred to, Dr. Harris has given a convenient method of computing *snz, cnz, dnz*. My friend, Dr. Virgil Snyder, has tabulated, under Professor Klein's direction, the Weierstrass sigma and zeta functions for the case $g_3=0$. The tables extend over nine parallelograms in the complex plane at intervals of one twenty-fourth of each period. They are now being published in Martin Schilling's 'Modell Verlag' (Halle). The case $g_2=0$ will next be treated.

I have also drawn the attention of the Section on former occasions to the importance of tabulating certain fundamental integrals, so as to increase our stock of what are called 'known functions,' in terms of which many other integrals might be expressed. Among these were the two integrals

$$\int_0^x \log \sin x dx,$$

$$\int_0^x J_0(x) dx.$$

In all that has been said I have confined myself to things that have been forced on my own attention. Many members of this Section and of its esteemed affiliated Society know of other standing problems. Not to go beyond the list of past officers that lies before me, I see the names of Eddy, Woodward, Waldo and Ziwet, who could tell us of the new problems in mechanics and dynamics; Gibbs, Hyde or Macfarlane could speak for quarternions and vector analysis; Bigelow for the mechanics of the atmosphere; Hayford for geodetic and tidal problems; Story for invariant theory; Johnson for differential equations; Moore for function theory; Beman, Phillips or Strong for geometry and analysis; Miller for group theory. Then to speak for the various fields of astronomical work we have a noble band consisting of Newcomb, Young, Pickering, Langley, Hall, Harkness, Hough, Van Vleck, Eastman, Stone, Chandler, Doolittle, Comstock, Paul, Upton, Holden, Kershner, Frisby, Barnard, Hall, Frost and Lord.

It would seem that the work of the Section not only advances science, but tends to prolong life; for I find only two starred names in the list of officers since the Section was reorganized on its present basis twenty years ago.

Rogers and Ferrel have entered into the larger life; and their works do follow them, for they are being carried on to wider issues.

JAMES McMAHON.

CORNELL UNIVERSITY.

* This paragraph has been amplified since the address was read.