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SCIENTIFIC BOOKS.

La géométrie Non-Euclidienne. Par P. Barbarin. Paris, C. Naud. Scientia, Février. 1902. Phys. Mathématique, No. 15. Pp. 79.

It is peculiarly appropriate that from Bordeaux, made sacred for non-Euclidean geometry by Hoüel, should emanate this beautiful little treatise, decorated with a 'gravure' reproducing part of a manuscript of Euclid, also with the official portrait of Lobachevski, but best of all, with a portrait of Riemann.

It begins from the hackneyed position: 'Experience therefore it is which has furnished to the ancient geometers a certain number of primitive notions, of axioms, or fundamental postulates put by them at the basis of the science.' But now we know there never was any pure receptivity. In all thinking enters a creative element. Every bit of experience is in part created by the subject said to receive it, but really in great part making it.

Professor Barbarin continues: 'From the epoch of Euclid, this number has been reduced to the strict minimum necessary, and all the others not comprised in this list, being capable of demonstration, are put in the class of theorems.' Now we know that Euclid omits to notice many of the assumptions he unconsciously employs, for example all the 'betweenness assumptions,' while Hilbert has at last rigorously demonstrated Euclid's assumption 'All right angles are equal,' and in turn one of Hilbert's assumptions has just been proved (see *Amer. Math. Monthly,* April, 1902, pp. 98–101).

The 'Elements' of Euclid, says Professor Barbarin, enjoyed throughout all the middle ages and still enjoy a celebrity that no other work of science has attained; this celebrity is due to their logical perfection, to the admirable concatenation of the propositions, and to the rigor of the demonstrations. 'Il mit dans son livre,' says Montucla, 'cet enchaînement si admiré par les amateurs de la rigueur géométrique.' "In vain," he adds, "divers geometers whom this arrangement has displeased, have attempted to better it.

"Their vain efforts have made clear how difficult it is to substitute for the chain made by the Greek geometer another as firm and as solid."

'This opinion of the historian of mathematics,' says our author, 'retains all its value even after the researches which geometers have undertaken for about a century to submit the fundamenal principles of the science to an acute and profound examination.' I add that the remarkable discoveries of Dehn (see SCIENCE, N. S., Vol. XIV., pp. 711–712) prove an unexpected superiority for Euclid over all successors down to our very day, and suggest the latest advance, which, though as yet unpublished, exists, for under date of April 2, 1902, Hilbert writes me: 'In einer andern Arbeit will ich die Lobatschefski'sche Geometrie in der ebene unabhangig von Archimedes begrunden.' That is, Hilbert will found Bolyai's geometry as he has Euclid's, without any continuity assumption.

To get the benefit of this brilliant achievement, I am holding back my own book on this fascinating subject.

Says Hilbert in his unpublished Vorlesung ueber Euklidische Geometrie, "The order of propositions is important. Mine differs strongly from that usual in text-books of elementary geometry; on the other hand, it greatly agrees with Euclid's order.

"So fuehren uns diese ganz modernen Untersuchungen dazu, den Scharfsinn dieses alten Mathematikers recht zu wuerdigen und aufs hoechst zu bewundern."

Again, *à propos* of Euclid's renowned parallel postulate, Hilbert says: "What sagacity, what penetration the setting up of this axiom required we best recognize if we look at the history of the axiom of parallels. As to Euclid hmself (circa 300 B. C.) he, *e. g.*, proves the theorem of the exterior angle before introducing the parallel axiom, a sign how deeply he had penetrated *in den Zusammenhang der geometrischen Saetze.*"

Professor Barbarin repeats the exploded error of attributing to Gauss the discovery of the non-Euclidean geometry in 1792. In the introduction to my translation of Bolyai's 'Science Absolute of Space,' pp. viii–ix, is a letter from Gauss, on which I there remark: "From this letter we clearly see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven."

In 1804 Gauss writes that in vain he still seeks the unloosing of this Gordian knot.

Again, with the date April 27, 1813, we read: "In the theory of parallels we are even now not farther than Euclid was. This is the 'partie honteuse' (shameful part) of mathematics, which soon or late must receive a wholly different form." Thus in 1813 there is in Göttingen still no light.

But in 1812 in Charkow, the non-Euclidean geometry already had been for the first time consciously created by Schweikart, whose summary characterization of it is given in SCIENCE, N. S., Vol. XII., pp. 842–846. This he communicated to Bessel and sent to Gerling and afterward to Gauss in 1818, so that it may claim to be the first *published* (not printed) treatise on non-Euclidean geometry.

By this time Gauss had progressed far enough to be willing to signify *privately* his acceptance of Schweikart's doctrines.

On p. 15, Barbarin makes a brief argument for Euclid's axiom, 'All right angles are equal.'

This argument was good before Hilbert and Veronese, since this axiom can never be proved by superposition. It is already a consequence of the assumptions preliminary to motion. This profounder analysis Barbarin has not attained to. He still uses as a postulate and supposes indispensable 'l'indeformabilité des figures en déplacement.' What Jules Andrade calls 'cette malheureuse et illogique définition' of Legendre, 'the shortest path between two points is a straight line,' Barbarin puts as an elementary proposition!

Manning also, p. 2, assumes it, thus invalidating and making ephemeral his pretty little 'Non-Euclidean Geometry' (Ginn & Co., 1901). Barbarin then proceeds to classify geometries by Saccheri's three hypotheses, the hypothesis of obtuse angle, the hypothesis of right angle, the hypothesis of acute angle, or that the angle sum of a rectilinear triangle is greater than, equal to, less than two right angles.

But the remarkable discoveries of Dehn have now shown that this classification is invalid.

Barbarin says, p. 16, 'Saccheri proves that the hypothesis of the obtuse angle is incompatible with postulate 6' of Euclid.

Dehn dissipates this supposed incompatibility by actually exhibiting a new geometry in which they amicably blend, which he calls the non-Legendrean geometry.

In the same way, the hypothesis of right angle amalgamates with the contradiction of Euclid's parallel-postulate in a geometry which Dehn calls semi-Euclidean. As Dehn states this result: There are non-Archimedian geometries in which the parallel-axiom is not valid and yet the angle-sum in every triangle is equal to two right angles. Thus the theorem (Legendre, 12th Ed., I., 23; Barbarin, p. 25): 'If the sum of the angles of every triangle is equal to two right angles the fifth postulate is true,' is seen to break down.

Manning's 'Non-Euclidean Geometry,' though it says (p. 93), 'The elliptic geometry was left to be discovered by Riemann,' gives only the single elliptic.

It never even mentions the double elliptic, or spherical or Riemannian geometry, which Killing maintains was the only form which ever came before Riemann's mind. If so, then Barbarin's book is like Riemann's mind. The Riemannian, as distinguished from the single elliptic, is the only form which appears in it. Killing was the first who (1879, *Crelle's Journal,* Bd. 83) made clear the difference between the Riemannian and the single elliptic space (or as he calls it, the polar form of the Riemannian).

Klein championed the single elliptic. Manning knows no other.

Professor Simon Newcomb, like Manning, deals only with the single elliptic in his treatise: 'Elementary theorems, relating to the geometry of a space of three dimensions and of uniform positive curvature *in the fourth dimension.*'

The last four words F. S. Woods replaces by seven dots in his article 'Space of constant curvature' (*Annals of Math.,* Vol. 3, p. 72), though blaming Professor E. S. Crawley for the error they contain.

Newcomb's also was the unfortunate conceit which dubbed this 'A Fairy-tale of Geometry,' a point of view from which he is still suffering in his latest little unburdening in *Harper's Magazine.*

Just so Lobachevski had the misfortune to call his creation 'Imaginary Geometry.'

Contrast John Bolyai's 'The Science Absolute of Space.'

In single elliptic space every complete straight line is of finite constant length πk.

Every pair of straight lines intersect and return again to their point of intersection, but have no other point in common.

In the so-called spherical space, that is the Riemannian space, two straight lines always meet in two points (opposites, or antipodal points) which are πk from each other.

The single elliptic makes the plane a unilateral or double surface, so that two antipodal points would correspond to one point, but to opposite sides of this one-sided plane with reference to surrounding three-dimensional elliptic space.

The geometry for two-dimensional Riemannian space coincides completely with pure spherics, that is with spherics established from postulates which make no reference to anything off of the sphere, inside or outside the sphere. Hence the great desirability of a treatise on pure spherics. It would at the same time be true and available for Euclidean and for Riemannian geometry.

Yet its relations to three-dimensional Euclidean and three-dimensional Riemannian space would differ radically.

Through every Riemannian straight line

passes an infinity of planes also Riemannian, and in each of these this straight has a determined and distinct center; but the straight is independent of the planes, and is defined by the postulates.

Now in the sphere the great circle and the one *pseudo*-plane which contains and fixes it, namely the sphere, are inseparable, since any portion, however minute, of either determines all the other as well as its center and radius.

In the single elliptic geometry the elliptic straight line does not divide the elliptic plane into two separated regions. We can pass from any one point of the plane to any other point without crossing a given straight in it. Starting from the point or intersection of two straights and passing along one of them a certain finite length, we come to the intersection point again without having crossed the other straight. Hence we can pass from what seems one side of the straight line to what seems the other without crossing it, that is, it is uni-lateral or double.

This single elliptic geometry is never mentioned in Barbarin's book; just as the Riemannian is never mentioned in Manning's book. First take your choice, then buy your non-Euclidean geometry.

On p. 36, Barbarin gives to Gauss the honor which belongs to Wallis of being the first to remark that the existence of unequal similar figures is equivalent, in continuous space, to the parallel postulate.

In Chapter VII., 'Les Contradicteurs de la géométrie non-euclidienne,' Professor Barbarin makes with unanswerable vigor the argument which I gave in my 'Report on Progress in Non-Euclidean Geometry' (SCIENCE, N. S., Vol. X., pp. 545–557).

There I quoted Whitehead who was the first to publish (March 10, 1898) "the extension of Bolyai's theorem by investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane.

"Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution.

"In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution.

"The same principles would enable the problem to be solved of the discovery in any kind of space of surfaces with their 'geodesic' geometry identical with that of planes in any other kind of space."

Now not only the strikingly important problem solved by Whitehead, but also the analogous problem indicated had both been solved by Barbarin and presented three months before to the Académie Royale de Belgique; but these investigations were only published after the appearance of my Report (October 20, 1899). They, as Barbarin says, p. 63, 'bring out in a striking manner the absolute independence of the three systems of geometry, which are able each to get everything from its own resources without need of borrowing anything from the others.' In each of the three spaces, Euclidean, Bolyaian, Riemannian, there exist surfaces whose geodesics have the metric properties of the straights of the two other spaces.

But the book in which these beautiful researches are published: 'Etudes de géométrie analytique non euclidienne par P. Barbarin, Bruxelles,' 1900, Hayez, pp. 168, has other titles to universal recognition.

Notwithstanding the ever-present example of Euclid, who never uses a construction or a figure which he has not shown to follow deductively from his two postulated figures, the straight and the circle, an insidious error crept into geometry, taught by Beman and Smith, who should know better, in the following words: (See their 'Geometry,' 1899, p. 70, § 112) "*Note on Assumed Constructions.*—It has been assumed that all constructions were made as required for the theorems.

"Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not yet been indicated, a regular heptagon has been mentioned, and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon, and to trisect any angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems." This is a fundamental blunder.

The construction so glibly assumed, to pass a circle through any three non-co-straight points, is equivalent to the assumption of the world-renowned parallel postulate, and thus has everything in the world to do with the sequence of the theorems. The assumed construction of a triangular from three sects which are to be its sides, by the method of Beman and Smith, p. 76, is equivalent to the assumption of the Archimedes postulate, which again has everything to do with the logical sequence of the theorems. In fact just this assumption makes ephemeral the beautiful method of Saccheri used in the book we are reviewing.

Hence we can appreciate that astounding achievement of Bolyai's young genius, his § 34, where he solves for his universe, Eu., I., 31. To draw a straight line through a given point parallel to a given straight line. His brilliant lead was followed more than half a century later by Gerard, but it is Barbarin who has ended the matter by deducing from certain very simple constructions of the trirectangular quadrilateral all the fundamental plane constructions.

In Chapter VIII. (La géometrie physique,' § 30 'La forme géométrique de notre univers') our author stresses the idea, that even if our universe were exactly Euclidean, it would be forever impossible for us to demonstrate this. As I said in my 'Non-Euclidean Geometry for Teachers,' p. 14, "If in the mechanics of the world independent of man we were absolutely certain that all therein is Euclidean and only Euclidean, then Darwinism would be disproved by the reductio ad absurdum. All our measurements are finite and approximate only. The mechanics of actual bodies in what Cayley called the external space of our experience, might conceivably be shown by merely approximate measurements to be non-Euclidean, just as a body might be shown to weigh more than two grams or less than two grams, though it never could be shown to weigh precisely, absolutely two grams."

Our author suggests the following experiment for proving our space non-Euclidean: From a point trace six rays sixty degrees apart. On them successively mark off the sects OA_0, OA_1, OA_2, $\cdots$, OA_n, of which each is the projection of the following. If we finish by finding between OA_n and 2^nOA_0 a difference of constant sense and greater than imputable to error of procedure, our universe is non-Euclidean.

In conclusion this beautiful little book has the advantage of being the production of an active and fertile original worker in the domain of which it treats. His 'Géométrie général des espaces' (1898), his 'Sur le paramètre de l'univers' and 'Sur la géométrie des êtres plans' (1901), 'Le cinquième livre de la métagéometrie,' (1901), 'Les cosegments et les volumes en géométrie non euclidienne' (1902), and his 'Poligones réguliers sphériques et non-euclidiens,' shortly to appear in that virile young monthly *Le Matematiche,* and which I had the advantage of reading in manuscript, show that Bordeaux is honored by a worthy successor of Hoüel, so universally beloved.

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Lamarck, The Founder of Evolution, His Life and Work, with Translations of His Writings on Organic Evolution. By ALPHEUS S. PACKARD, M.D., LL.D. New York, London and Bombay, Longmans, Green & Co. 1901. Pp. xii+451.

This appears to the reviewer to be a noteworthy book; he has read it from cover to cover with so much pleasure that he ventures to predict that it will prove a source of satisfaction to that large body of readers who are interested in the rise of evolutionary thought.

Larmarck lived in advance of his age and died comparatively unappreciated.

Although quiet and uneventful, his life was a busy one, and, as sketched by Dr. Packard, his noble character, his generous disposition and his deep intellectuality are well brought out.

His devoted and loyal daughter, Cornelie, without whose assistance his later works could