varying more than 10 per cent. The gradient at such places is 8 to 10 feet per mile.

Professor Schlicter's paper was discussed by Professors C. R. Van Hise, J. B. Johnson, F. W. King, F. E. Turneaure and E. A. Birge.

Mr. Pfund discussed the 'Dispersion and Absorption of Selenium.' By devising a new method for depositing films of the aniline dyes on glass and for photographing the interference fringes produced by a Michelson interferometer, the dispersion of amorphous selenium, a comparatively opaque substance, has been successfully studied. The refractive index of selenium rises with extraordinary rapidity until at the limit of the photographic field it reaches a value of 3.13, one of the very highest known. In general, the light-absorbing power of selenium lies between that of the aniline dyes and that of the metals. With a small concave grating, it has been found that selenium absorbs light more and more strongly as the end of the ultra-violet spectrum is approached, instead of there being a region of retransmission.

C. K. LEITH.

DISCUSSION AND CORRESPONDENCE.

THE MATHEMATICAL THEORY OF THE TOP, SIMPLIFIED.

TO THE EDITOR OF SCIENCE: Professor A. G. Greenhill has been good enough to show me his terse method of treating the top integrals. As this is a subject on which Professor Greenhill speaks authoritatively, and will interest a number of your readers, in particular his many friends in Sections A and B of the American Association, I suggest that it be published in SCIENCE.

CARL BARUS.

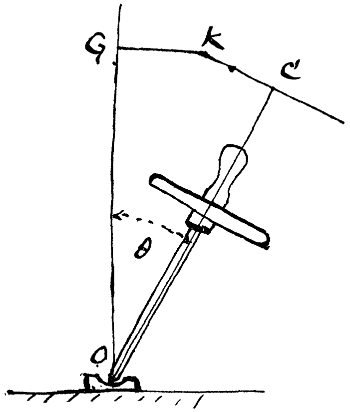
Brown University, Providence R. I.,
November 20, 1901.

Let the vector OH represent the resultant angular momentum of a symmetrical top; spinning about its point O is a small smooth fixed cup, as in the Maxwell top.

Since the axis Og of the torque of gravity is always horizontal H will describe a curve (a Poinsot herpolhode), in a fixed horizontal plane at a height OG above O, the vertical vector OG representing the constant component G of angular momentum about the vertical.

We assume that the component G' of the angular momentum of the top about its axis OC remains constant, as there is nothing to alter it, if the top is symmetrical.

FIG. 1.



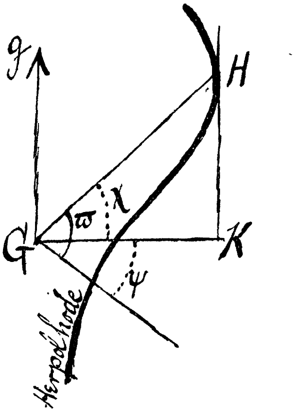
Expressed by Euler's angles θ and ψ the vector OH has the components (Figs. 1 and 2)

$$(1) \quad OC = G', \quad CK = A_1 \sin\vartheta \frac{dx}{dt}, \quad KH = A_1 \frac{d\vartheta}{dt},$$

A_1 denoting the moment of inertia of the top about an axis through O perpendicular to OC.

The velocity of H is equal to the torque of gravity $Wgh \sin\vartheta$, so that, denoting the polar

FIG. 2.



coordinates of H in the horizontal plane GHK by ρ and π, and resolving in the radial direction GH.

$$(2) \quad \frac{d\rho}{dt} = Wgh \sin\vartheta \cos GHK = Wgh \sin\vartheta \frac{KH}{\rho}$$

(3) $$\rho \frac{d\rho}{dt} = A_1 Wgh \sin \vartheta \frac{d\vartheta}{dt},$$

and integrating,

(4) $$\tfrac{1}{2}\rho^2 = A_1 Wgh (E - \cos \vartheta).$$

To make the equations homogeneous, put

(5) $$OG = \delta,\ OC = \delta^1,\ 4A_1 Wgh = k^2,$$

so that

(6) $$\rho^2 = \tfrac{1}{2}k^2(E - \cos \vartheta).$$

Now denoting the perpendicular GK from G on the tangent at H by p,

(7) $$p \sin \vartheta = \delta' - \delta \cos \vartheta,$$

so that, eliminating ϑ,

(8) $$p^2 = \frac{[\delta' k^2 - \delta(Ek^2 - 2\rho^2)]^2}{k^4 - (Ek^2 - 2\rho^2)^2},$$

and this is the characteristic geometrical property of a Poinsot herpolhode; it also defines the trace of a rolling line of curvature, the intersection of an ellipsoid and a confocal hyperboloid of two sheets, and then δ is the angle between the generating lines of the confocal hyperboloid* of one sheet through H (Darboux).

Since

(9) $$KH^2 = GH^2 - GK^2,$$

(10) $$A_1^2\left(\frac{d\vartheta}{dt}\right)^2 = 2A_1 Wgh(E - \cos\vartheta) - \frac{(G' - G\cos\vartheta)^2}{\sin^2\vartheta};$$

and putting $\cos\vartheta = z$, $\frac{Wgh}{A_1} = n^2$,

(11) $$\left(\frac{dz}{dt}\right)^2 = 2n^2 Z,$$

(12) $$Z = (E - z)(1 - z^2) - \frac{(G' - Gz)^2}{2A_1 Wgh} = (E - z)(1 - z^2) - 2\left(\frac{\delta' - \delta z}{k}\right)^2.$$

Denoting the roots of $Z = 0$ by z_1, z_2, z_3, and arranging them so that

(13) $$z_1 > 1 > z_2 > z > z_3 > -1,$$

then with $\frac{d\vartheta}{dt}$ positive, $\frac{dz}{dt}$ negative, as in Fig. 1,

(14) $$nt = \int_z^{z_2} \frac{dz}{\sqrt{2Z}},$$

an elliptic integral of the first kind; and, by inversion, z is an elliptic function of t.

To make the reduction to Legendre's standard form, put

* *Proc. London Math. Society,* XXVI., XXVII.

(15) $$z = z_2 \sin^2\phi + z_3 \cos^2\phi,$$

(16) $$z - z_3 = (z_2 - z_3)\sin^2\phi,$$

(17) $$z_2 - z = (z_2 - z_3)\cos^2\phi,$$

(18) $$z_1 - z = (z_1 - z_3)\Delta^2\phi,$$

(19) $$k^2 = \frac{z_2 - z_3}{z_1 - z_3},\ k'^2 = \frac{z_1 - z_2}{z_1 - z_3};$$

then

(20) $$nt = \sqrt{\left(\frac{2}{z_1 - z_3}\right)} \int_\phi^{\frac{1}{2}\pi} \frac{d\phi}{\Delta\phi} = \sqrt{\left(\frac{2}{z_1 - z_3}\right)}(K - F\phi),$$

(21) $$F\phi = K - mt,\ mt = \sqrt{\left(\frac{z_1 - z_3}{2}\right)}\, nt.$$

Then, in Jacobi's notation,

(22) $$\phi = am(K - mt),$$

and in Gudermann's notation,

(23) $$z = z_2 Sn^2(K - mt) + z_3 Cn^2(K - mt),$$

the expression of z or $\cos\vartheta$ by elliptic functions of t.

Next, denoting the angle KGH by χ,

(24) $$\tan\chi = \frac{KH}{GK} = \frac{A_1 \sin\vartheta \frac{d\vartheta}{dt}}{GK\sin\vartheta} = \frac{\sqrt{(2A_1 WghZ)}}{G' - G_2} = \frac{\sqrt{2Z}}{2\frac{\delta' - \delta_2}{k}},$$

(25) $$\sin\vartheta\cos\chi = \frac{G' - G_2}{\sqrt{[2A_1 Wgh(E - z)]}},$$

$$\sin\vartheta\sin\chi = \frac{\sqrt{Z}}{\sqrt{(E - z)}}.$$

Resolving transversely to GH, or rather, taking the moment of the velocity of H round G,

(26) $$\rho^2\frac{d\pi}{dt} = Og \cdot GK = Wgh\sin\vartheta \cdot GK = Wgh(G' - G\cos\vartheta)$$

so that, from (6),

(27) $$\frac{d\pi}{dt} = \frac{G' - G^2}{2A_1(E - z)}$$

(28) $$\pi = \frac{Gt}{2A_1} + \frac{G' - GE}{2A_1}\int\frac{dt}{E - z} = \frac{\delta}{k}nt + \frac{\delta' - \delta E}{k}\int_z^{z^2}\frac{dz}{(E - 2)\sqrt{2Z}},$$

involving an elliptic integral of the III. kind, and then

(29) $$\psi = \pi - \chi.$$

But the component of the angular momentum round the vertical

$$(30)\qquad OC\cos\vartheta + CK\sin\vartheta = G'\cos\vartheta + A_1\sin^2\vartheta\frac{d\psi}{dt} = G$$

so that

$$(31)\qquad \frac{d\psi}{dt} = \frac{G - G'\cos\vartheta}{A_1\sin^2\vartheta} = \frac{G - G'z}{A_1(1-z^2)} = \frac{G-G'}{2A_1}\cdot\frac{1}{1-z} + \frac{G+G'}{2A_1}\cdot\frac{1}{1+z},$$

which gives ψ as the sum of two elliptic integrals of the III. kind, their addition into a single integral (Legendre) is shown by (28), (29).

The reduction to the Weierstrassian form is effected by putting

$$(32)\qquad p\mu - e_a = s - s_a = \tfrac{1}{2}M^2(z - z_a),$$

where M is a homogeneity factor at our disposal; and now

$$(33)\qquad nt = \int_s^{s_2}\frac{Mds}{\sqrt{S}},\quad S = 4(s-s_1)(s-s_2),(s-s_3),$$

$$(34)\qquad p'^2\mu = S = \tfrac{1}{2}M^6 Z,$$

$$(35)\qquad \frac{nt}{M} = \int_s^\infty\frac{ds}{\sqrt{S}} - \int_{s_2}^\infty\frac{ds}{\sqrt{S}} = \mu - \omega_2.$$

If v, σ, Σ denote the corresponding values of μ, s, S when $z = E$,

$$(36)\qquad pv - p\mu = \tfrac{1}{2}M^2(E - z)$$

$$(37)\qquad pv - e_a = \sigma - s_a = \tfrac{1}{2}M^2(E - z_a)$$

$$(38)\qquad ip'v = -\sqrt{(-\Sigma)} = M^3\frac{\delta' - \delta E,}{k}$$

$$(39)\qquad z_1 > E > z_2 > z > z_3,\quad s_1 > \sigma > s_2 > s > s_3,$$

so that

$$v = \omega_1 + \psi\omega_3\,;$$

and (28) becomes

$$(40)\qquad \pi = \frac{\delta}{k}nt + \tfrac{1}{2}\int\frac{ip'vdn}{pv - p\mu},$$

with the elliptic integral of the III. kind in the standard form of Weierstrass.

In the steady motion of the top, $\frac{d\vartheta}{dt} = 0$, $\frac{d\psi}{dt} = \mu$, a constant, H and K coincide, and

$$(41)\qquad \rho = GK = OC\sin\vartheta - CK\sin\vartheta = G'\sin\vartheta - A_1\mu\sin\vartheta\cos\vartheta,$$

$$(42)\qquad Wgh\sin\vartheta = \rho\mu = \sin\vartheta\,(G'\mu - A_1\mu^2\cos\vartheta),$$

and dropping the factor $\sin\vartheta$,

$$(43)\qquad A_1\mu^2\cos\vartheta - G'\mu + Wgh = 0.$$

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CURRENT NOTES ON PHYSIOGRAPHY.

NEW ORDNANCE MAP OF ENGLAND.

THE 'Colored One-inch Map of England,' now in process of publication, marks a great improvement on both the old and the new series of the earlier ordnance survey inch-to-a mile maps. Relief is shown by brown hachures, drawn with much accuracy of expression, and by red contours at intervals of 100 feet. Water is in blue, with blue contours in the sea for every 25 feet of depth near shore; the chief roads are in ochre, woodland on some of the sheets is green, and culture is black. Most of the sheets represent quadrangles measuring 18 miles east and west by 12 miles north and south; but for southern England the sheets frequently include larger areas, according to some system that is not immediately apparent. Some 130 sheets have now been published, the standard sheets costing a shilling each. While looking over them, local geographical features are brought vividly to mind. The Falmouth sheet, Cornwall, includes the typical drowned valleys of Fal and Helston rivers, open to the sea on the east side of the even uplands back of the Lizard, and of Loe river, closed on the more exposed western coast by Portleven sands, one of the few beaches of this ragged shore line. The contours along the valley sides here and on the neighboring Ivybridge and Boscastel sheets are of much smoother curvature than those that follow the coast, thus showing that the shore line in this district of resistant ancient rocks is in that immature stage of development when its irregularity of detail has become greater than it was in the initial stage. Where the coast consists of weaker Mesozoic rocks, as shown on the Exeter and Sidmouth sheets, a smoother shore line of greater retreat and more mature express sion is found. A little further east, where the Bridport and Weymouth sheets join, the long sweeping curve of Chesil bank is finely displayed.

The relations of rivers to their valleys offer some interesting problems. In certain meandering valleys the rivers sweep around the valley curves in a most competent fashion, pressing against their outer banks and demanding an increased breadth of meander belt; the Torridge in Devonshire (Chulmleigh sheet) and the